A PROOF AND EXTENSION OF THE JORDAN-BROUWER SEPARATION THEOREM*

BY

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1. The theorem on the separation of n-space by an $(n-1)$-dimensional manifold† suggests the following more general problem of analysis situs.

Given a figure C of known connectivity immersed in an n-space H, what can be said about the connectivity of the domain $H-C$ residual to C?

It will be shown that a certain duality exists between the topological invariants of C and $H-C$, and that when C is an $(n-1)$-dimensional manifold the separation theorem is merely one aspect of this duality. The paper also touches upon a number of well known related questions,—among them, the invariance of dimensionality and regionality, the approachability of points of C from $H-C$, the invariance of the topological constants of C and $H-C$, and so on.

Of course, the main difficulties in such problems as the above are of a point-theoretic order. They all yield, however, to simple pinching processes, except for the use of which the following treatment will be purely combinatorial. The earlier sections, §§ 2–8, are expository and give a rapid though essentially complete survey of the terminology and combinatorial machinery needed in the sequel. The fundamental part of the discussion, with illustrations and applications, is really all contained in §§ 9–12.

The theory of connectivity may be approached from two different angles depending on whether or not the notion of *sense* is developed and taken into consideration. We have adopted the second and somewhat simpler point of view in this discussion in order to condense the necessary preliminaries as much as possible. A treatment involving the idea of sense would be somewhat more complicated but would follow along much the same lines.

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* Based on a paper presented to the Society, April 29, 1916.

† First completely proved for the case $n=1$ (Jordan's theorem) by O. Veblen, these Transactions, vol. 6 (1905), p. 83, cf. addendum, vol. 14 (1913), p. 65 (second footnote). Proved for a general n by L. E. J. Brouwer, Mathematische Annalen, vol. 71 (1912), p. 314. A proof of Jordan's theorem along lines similar to those used in this paper was given by the author in the Annals of Mathematics, vol. 21 (1920), p. 180.

Chains and their combinatorial properties

2. Certain advantages of symmetry are gained by setting the problem in the space of n-dimensional spherical geometry rather than in euclidean n-space. We shall therefore take as our fundamental domain the n-sphere

$$H^n: \qquad x_0^2 + x_1^2 + \cdots + x_n^2 = 1$$

in the space of $n + 1$ real variables. The geodesics (great circles) determined on the n-sphere H^n by its intersection with $n - 1$ linearly independent n-planes through the origin play the rôle in this geometry of the straight lines in ordinary euclidean space. A region will be said to be *convex* if any two of its points may be joined by one and only one geodesic arc made up of points of the region.

An n-plane through the origin subdivides the n-sphere H^n into a pair of n-regions bounded and separated by an $(n - 1)$-sphere. The latter may in turn be subdivided in the same way into a pair of $(n - 1)$-regions separated by an $(n - 2)$-sphere, and so on down to a pair of 0-regions, or points. The resulting partition of the n-sphere, consisting of two regions of every dimensionality from 0 to n, will be called an *elementary subdivision* of H^n. It is evident that any k-region of an elementary subdivision may be cut up by an n-plane through the origin into a pair of convex k-regions separated by a convex $(k - 1)$-region and that by repeating this process of repartitioning, the n-sphere may be cut up into arbitrarily small convex regions of dimensionalities 0 to n. These regions will be called k-cells, where k denotes dimensionality.

If the repartitioning is done in a perfectly random fashion, there is nothing to prevent the boundary of a k-cell from containing a part but not all of a cell of lower dimensionality. It is then possible to carry the repartitioning still further, beginning with the boundaries of the n-cells and working down to the boundaries of the 1-cells, until, finally, the boundary of each cell consists of complete cells only. The resulting collection of cells will be called a *subdivision* of the n-sphere H^n.

A subdivision S' will be said to be *derived* from a subdivision S if it can be obtained from S without the use of any other operations than the repartitioning of cells in the manner just described. Thus, by definition, every subdivision is derived from an elementary one. It should be observed that the operations of repartitioning may always be performed in such an order that all intermediate figures will themselves be subdivisions. This would be the case, for example, if the cells were repartitioned in such an order that no one of them was ever touched until the cutting up of its boundary had been completed.

3. There is, of course, no difficulty in writing out explicitly the analytical expressions that determine a cell. They consist merely of the equation of H^n taken in conjunction with certain linear equalities and inequalities depending on

the n-planes that cut out the cell. From the form of these expressions, it follows at once that every cell is convex. Consequently, any two i-cells are *homeomorphic* with one another (i.e., in point-for-point continuous correspondence), and the boundary of any i-cell is homeomorphic with an $(i-1)$-sphere, since the latter bounds each of two i-cells on an i-sphere.

4. Any set of cells from the same subdivision of H^n will be called a *chain* provided the set never contains a cell E without also containing all cells on the boundary of E. A chain may therefore be a very mixed agglomeration of cells. If, however, it consists only of i-cells and the cells of their boundaries, it will be called an *i-chain.*

The simplest i-chain is one containing a single i-cell. It will be called a *cellular i-chain.* Any i-chain will be said to be the *sum modulo* 2 of the cellular i-chains determined by its individual i-cells; in symbols,

$$K^i = K_1^i + K_2^i + \cdots + K_s^i \qquad \text{(mod. 2)} \tag{1}$$

We shall also speak of the sum of two or more arbitrary i-chains of the subdivision, and it is here that the *modulo* 2 feature of the operation first comes into evidence. To form the sum, we express each i-chain in terms of its cellular components, as in (1), add components, and reduce coefficients modulo 2. In other words, the sum-chain contains the i-cells that belong to an odd number of chains of the sum, but no others.

5. An i-chain K^i $(i > 0)$ will be said to be *closed* if each of its $(i-1)$-cells belongs to the boundary of an even number of its i-cells; otherwise, it will be called *open*, or *bounded*, and its boundary K^{i-1} will be the $(i-1)$-chain determined by such $(i-1)$-cells as belong to the boundary of an odd number of i-cells of K^i. It will be convenient to express the relation of K^{i-1} to K^i symbolically by the notation (adapted from the congruences of Poincaré)

$$K^i \equiv K^{i-1} \qquad \text{(mod 2)} \tag{2}$$

which may be read "K^i is bounded by K^{i-1}". The expression

$$K^i \equiv 0 \qquad \text{(mod 2)}$$

signifies that K^i has no boundary and is therefore closed. We shall frequently condense the notations (1) and (2) by omitting to write mod 2.

A 0-chain will be open or closed according as it consists of an odd or an even number of points.

The boundary K^{i-1} of an i-cell is a simple illustration of a closed $(i-1)$-chain, since every $(i-2)$-cell of K^{i-1} belongs to precisely two $(i-1)$-cells of K^{i-1}.

Now, the relations

$$K_s^i \equiv K_s^{i-1} \qquad (s = 1, 2, \ldots, t) \qquad \pmod 2$$

evidently imply

$$\sum_s K_s^i \equiv \sum_s K_s^{i-1} \qquad \pmod 2$$

or, in words, the boundary of a sum of i-chains is the sum of the boundaries of the i-chains themselves. Thus, in particular, the sum of two or more closed i-chains is itself closed, when it does not vanish. It also follows that the boundary of an open $(i + 1)$-chain is a closed i-chain, since it is the sum of the boundaries of the individual $(i + 1)$-cells of the $(i + 1)$-chain.

6. We proceed to define the connectivity numbers of a chain C. Let K^i be an i-chain of C, that is to say, a chain composed of cells of C. Then, by a second adaptation from Poincaré, we shall write

$$(3) \qquad K^i \sim 0 \qquad (\mathrm{mod}\ 2,\ C)$$

(K^i is *homologous to zero*, or *bounds* on C) provided K^i is the boundary of some open $(i + 1)$-chain of C. The relations $K_1^i \sim 0$ and $K_2^i \sim 0$ evidently imply $K_1^i + K_2^i \sim 0$; therefore, it will be legitimate to operate with homologies as though they were linear equations modulo 2. The expression $K_1^i \sim K_2^i$ will, of course, be just another way of writing $K_1^i + K_2^i \sim 0$. We denote by $R^i - 1$ the maximum number of closed non-bounding i-chains $K_1^i, K_2^i, \ldots,$ of C that are independent with respect to homologies; that is, such that there exists no relation between the chains K_s^i of the form

$$\sum_{s=1}^{R^i-1} \epsilon_s K_s^i \sim 0 \qquad (\mathrm{mod}\ 2,\ C)$$

unless the coefficients ϵ_s are all zero, and such that every other closed i-chain K^i of C is related to the chains K_s^i by an homology

$$K^i \sim \sum \epsilon_s K_s^i \qquad (\mathrm{mod}\ 2,\ C).$$

The number R^i is called the *ith connectivity number* of C.* It will be seen later on that R^i is not only an invariant of C but also of the set of points determined by C. For the moment, it will be sufficient to observe that R^0 denotes the number of separate connected parts of C. It is sometimes advantageous to

* The numbers R^i are the modulo 2 analogues of the Betti numbers of Poincaré. They were first introduced in a paper by Professor Veblen and the author, Annals of Mathematics, vol. 14 (1913), p. 163.

consider connectivity numbers of higher dimensionalities those that of any cell of C. Such numbers are automatically unity, from their definition.

The connectivity number R^i of C which is of the same dimensionality i as the cells of highest dimensionality appearing in C satisfies a relation that we shall now recall for future reference. Let there be α i-cells

$$A_1^i, A_2^i, \ldots, A_\alpha^i$$

in C, and let the symbols associated with these i-cells be regarded as variables free to take on either of the values 0 or 1. Then, to every choice of a set of values for these variables such that at least one variable is not zero, there may be associated an i-chain of C determined by the i-cells with symbols unity. Conversely, by reversing the process, to every i-chain of C there may be associated a set of values of the variables. Now, if $i > 0$, let us write a modular equation

$$E_q: \qquad \sum_p \epsilon_{qp} A_p^i \equiv 0 \pmod{2}$$

corresponding to each $(i-1)$-cell A_q^{i-1} of C, where the coefficient ϵ_{qp} has the value unity or zero according as the cell A_q^{i-1} is or is not on the boundary of the cell A_p. Then, to any set of values of the variables A_p^i satisfying the simultaneous equations E_q, there will be associated a *closed* i-chain of C, and conversely. For, among the variables with non-vanishing coefficients appearing in each equation, there must be an even number or zero that have the value unity, which means that each $(i-1)$-cell of C is on the boundary of an even number or none of the i-cells of the i-chain determined by the solution of the equation E_q. Thus, if ρ be the number of linearly independent equations E_q, the maximum number of independent solutions must be

$$R^i - 1 = \alpha - \rho,$$

which is the relation we set out to find.

If $i = 0$, there are no equations E_q. For this special case, we evidently have

$$R^0 - 1 = \alpha - 1.$$

We say that a closed n-chain is *irreducible* if its invariant R^n has the value 2, that is, if the chain is not the sum of two or more closed sub-chains. The points of an irreducible closed n-chain form an *n-dimensional manifold*.

7. Just as in the case of subdivisions, §2, a chain C' will be said to be *derived* from a chain C if it is one of a sequence of chains beginning with C and such that each member of the sequence is transformable into the next one by partitioning a single k-cell E^k into a pair of k-cells F_1^k and F_2^k separated by a $(k-1)$-cell F^{k-1}. The invariants of the chain C' are the same as those of C, for, when

the cell E^k is cut, the only i-chains to appear in the figure that are not mere subdivisions of old ones are the $(k-1)$-chains containing F^{k-1} and the k-chains containing one but not both of F_1^k and F_2^k. The latter chains are open and therefore do not increase the connectivity numbers; nor do they decrease them by setting up new relations of bounding among old $(k-1)$-chains, for their boundaries all contain the new cell F^{k-1}. The $(k-1)$-chains containing F^{k-1}, whether open or closed, are transformed into old chains which no longer contain F^{k-1} by the addition of the boundary of F^k; therefore, they can have no effect on the connectivity numbers, one way or the other. Thus, we pass from C to C' by a series of operations which do not alter the connectivity numbers.

THEOREM S^n. *The connectivity numbers of any subdivision S^n of an n-sphere are all unity except the nth one which has the value* 2.

For, by §2, the subdivision S^n is derivable from an elementary subdivision and therefore has the same connectivity numbers as the latter. But in an elementary subdivision, every closed chain bounds a cell, with the exception of the n-chain determined by the two n-cells.

COROLLARY. *Any closed $(n-1)$-chain K^{n-1} of the subdivision S^n of the n-sphere bounds exactly two open n-chains. Moreover, these two n-chains have only the points of K^{n-1} in common.**

For, since the number R^{n-1} is unity, there must exist an open n-chain K^n such that

$$K^n \equiv K^{n-1} \qquad \text{(mod 2, } C),$$

and consequently a second n-chain $K^n + S^n$, determined by the n-cells of S^n which do not belong to K^n, such that

$$K^n + S^n \equiv K^{n-1} \qquad \text{(mod 2, } C)$$

and such that it has in common with K^n only the points of K^{n-1}.

Now, if there were a third n-chain L^n bounded by K^{n-1}, there would be two independent closed n-chains

$$L^n + K^n \equiv 0 \text{ and } L^n + K^n + S^n \equiv 0,$$

and the number R^n would be at least 3, contrary to the theorem.

8. Let $\bar{H}^m$ and H^n be an m- and an n-sphere respectively, and let $\bar{C}$ be any chain of a subdivision of $\bar{H}^m$. Then, by an extension of terminology, we shall

* This is a somewhat weakened form of the theorem on the separation of n-space by a generalized polyhedron. For a proof of the latter theorem making use of modulo 2 equations, see O. Veblen, these Transactions, vol. 14 (1913), p. 65, and vol. 15 (1914), p. 506.

speak of any set of points C of H^n in reciprocal one-one continuous correspondence with $\bar{C}$ as a *chain immersed* in H^n. For example, if $\bar{C}$ is the boundary of a 2-cell, C may be any simple closed curve of H^n. The *cells* of the chain C will be the images in H^n of the cells of $\bar{C}$, so that C and $\bar{C}$ will both have the same cellular structure and, consequently, the same connectivity numbers. We shall frequently make use of the fact that there exists a derived chain of C made up of arbitrarily small cells. That such a chain does exist follows at once from the uniform continuity of the correspondence between the closed sets of points determined by C and $\bar{C}$, for we know that the cells of $\bar{C}$ may be redivided to any degree of smallness.

If the chain C does not fill up the entire space H^n, the residual part of H^n will form a certain domain $H^n - C$ made up of inner points. We proceed to define the connectivity numbers of this domain. Any chain of any subdivision of H^n will be called a chain of $H^n - C$ provided it is wholly contained in $H^n - C$. Among the chains of $H^n - C$ will be set up the following homologies: (1) Each closed i-chain will be said to be homologous to its derived chains; (2) each closed i-chain which bounds an open $(i+1)$-chain of $H^n - C$ will be said to be homologous to zero. We combine homologies (1) and (2) like linear equations modulo 2 and denote by $(\bar{R}^i - 1)$ the maximum number of linearly independent closed i-chains of $H^n - C$. *A priori*, there is no reason why the number $\bar{R}^i$ should be finite in this case, since we are now dealing with equations in an infinite number of variables. It will be proved further on, however, that the numbers $\bar{R}^i$ are all finite and also pure topological invariants of the domain $H^n - C$, in spite of the fact that a metric on H^n has been used in defining them. The number $\bar{R}^0$ is of particular importance and evidently denotes the number of separate connected regions in $H^n - C$.

Since we shall only be concerned with the relations between chains under homologies, it will be legitimate to do away with the distinction between a chain of $H^n - C$ and its derived chains. We shall therefore regard any two chains with a common derived chain as equivalent chains, to be denoted by the same symbol K^i. A closed i-chain will then be said to bound if it bounds in any of its derived forms, so that the terms *bounding* and *homologous to zero* will henceforth be synonymous.

ON THE DUAL CONNECTIVITIES OF C AND $H^n - C$

9. We now come to the body of the discussion.

THEOREM T^i. *Let C^i be a cellular i-chain (§4) immersed in the n-sphere H^n. Then, the connectivity numbers of the domain $H^n - C^i$ residual to C^i are all unity. In other words, every closed chain L^k of $H^n - C^i$ bounds.*

The theorem is trivial if i is zero, in which case, C^i reduces to a point C^0.

For there are no closed chains in $H^n - C^i$ of dimensionality greater than or equal to n. Moreover, by the corollary to Theorem S^n, every closed $(n-1)$-chain of $H^n - C^0$ bounds twice in H^n and therefore once in $H^n - C^0$. Every closed chain of lower dimensionality bounds as often as we please in $H^n - C^0$.

The general case will be handled by induction with respect to i. We shall assume the validity of Theorem T^{i-1} and first prove a lemma.

LEMMA U^i. *Let the cellular i-chain C^i be subdivided into two cellular i-chains A and B, respectively, meeting in a cellular $(i-1)$-chain C^{i-1}. Then every k-chain L^k of $H^n - C^i$ which bounds both in $H^n - A$ and $H^n - B$ must also bound in $H^n - C^i$.*

If $k = n-1$, the chain L^k bounds exactly two open n-chains of H^n meeting in L^k, by the corollary of § 7. The connected set of points $C^i = A + B$ must therefore lie wholly within one of these two open n-chains, since it does not meet L^k. Consequently, the other n-chain lies in $H^n - C^i$, and L^k bounds in this region.

Now, suppose that $k < n-1$, so that there exist chains in H^n of dimensionalities as high as $k+2$. By hypothesis, there exist two open chains L^{k+1}_A and L^{k+1}_B such that

$$(4)\qquad \begin{aligned} L^{k+1}_A &\equiv L^k \qquad (\text{mod } 2,\ H^n - A) \\ L^{k+1}_B &\equiv L^k \qquad (\text{mod } 2,\ H^n - B) \end{aligned}$$

and these combine to form the closed chain

$$L^{k+1}_A + L^{k+1}_B \equiv 0 \qquad (\text{mod } 2,\ H^n - C^{i-1}),$$

as illustrated in the figure which is purely schematic. We shall assume that L^{k+1}_A and L^{k+1}_B meet B and A, respectively, otherwise the lemma would be true without further argument.

Now, by Theorem T^{i-1}, which we are assuming in the induction, there exists an open $(k+2)$-chain M^{k+2} such that

$$M^{k+2} \equiv L^{k+1}_A + L^{k+1}_B \qquad (\text{mod } 2,\ H^n - C^{i-1}).$$

This chain cuts the chains A and B in mutually exclusive closed sets of points and may therefore be broken up into cellular $(k+2)$-chains so small that no one of them meets both A and B, since there is a definite interval of separation between two non-overlapping closed sets of points. Now, let $\overline{M}^{k+2}$ be the sum of the cellular $(k+2)$-chains of M^{k+2} that meet A, and therefore not B, and let $\overline{L}^{k+1}$ be the boundary of $\overline{M}^{k+2}$. Then we have

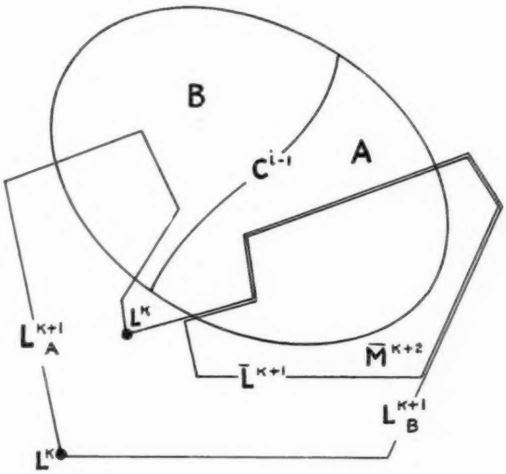
$$(5)\qquad M^{k+2} + \overline{M}^{k+2} \equiv (L^{k+1}_B + \overline{L}^{k+1}) + L^{k+1}_A \equiv 0 \ (\text{mod } 2,\ H^n - C^{i-1}).$$

But $L^{k+1}_B + \bar{L}^{k+1}$ meets neither A nor B and therefore lies in $H^n - C^i$. Hence, by (5) and the first relation in (4),

$$L^{k+1}_B + \bar{L}^{k+1} \equiv L^k \qquad (\text{mod } 2,\ H^n - C^i),$$

which establishes the lemma.

Theorem T^i now follows at once by the ordinary pinching process. If the theorem were false for the chain C^i, it would be false for one of two cellular sub-chains, by the lemma, and by repeating the argument it would be possible to find a sequence of sub-chains C^i_s of C^i closing down upon a single point C^0 and for each of which the theorem would be false. But, by Theorem T^0, every closed k-chain L^k of $H^n - C^i$, and therefore of $H^n - C^0$, would bound a chain



L^{k+1} of H^n which did not meet C^0 and which therefore could not meet all of the chains C^i_s converging on that point. Therefore, the theorem must be true, since the assumption that it is false leads to a contradiction.

COROLLARY V^i. *A cellular i-chain immersed in an n-sphere H^n cannot fill H^n.*

For let C^i be broken up into two cellular parts, A and B, as in Lemma U^i, and let P_A and P_B be points of A, and B, respectively but not of the chain C^{i-1} common to A and B. Then, by Theorem T^{i-1}, the 0-chain $P_A + P_B$ bounds a 1-chain in $H^n - C^{i-1}$ which must contain a broken line of geodesics connecting P_A with P_B. But this broken line meets A and B in mutually exclusive closed

sets of points and must therefore contain points that belong to neither of these sets. Such points must be points of $H^n - C^i$.

COROLLARY W^i. *Let C be the sum and C^{i-1} the intersection of two closed sets of points A and B. Then every closed k-chain L^k $(k<n-1)$ of $H^n - C$ which bounds a chain L_A^{k+1} of $H^n - A$ and a chain L_B^{k+1} of $H^n - B$ must also bound in $H^n - C$ provided the chains L_A^{k+1} and L_B^{k+1} may be so chosen that $L_A^{k+1} + L_B^{k+1}$ bounds in $H^n - C^{i-1}$. Moreover, the corollary is valid even if $k = n-1$ unless C^{i-1} is the null set.*

For the proof of Lemma U^i is applicable here with scarcely a change.

10. We are now in a position to prove the duality theorem mentioned in the introduction. In order to separate out the difficulties, however, let us first consider an important special case which admits of a simpler proof than the general one.

THEOREM X^i. *Let C^i be an i-sphere immersed in an n-sphere H^n. Then the connectivity numbers R^s of C^i are related to the connectivity numbers $\bar{R}^s$ of the residual space $H^n - C^i$ by the equations*

$$R^i = \bar{R}^{n-i-1} = 2, \quad R^s = \bar{R}^{n-s-1} = 1 \qquad (s \neq i).$$

The theorem states, in other words, that there exists but one independent closed non-bounding chain in $H^n - C^i$. This chain will be of dimensionality $(n - i - 1)$. It will be said to *link* the i-sphere C^i.

If $i = 0$, the i-sphere C^i is a pair of points, so that the theorem is both trivial and obvious. The $(n-1)$-chain linking C^i is any closed $(n-1)$-chain of L^{n-1} such that one of the points of C^i lies in each of the two open n-chains bounded by L^{n-1} in H^n, (§7). All closed chains of lower dimensionalities bound as often as we please in $H^n - C^i$.

The case $i>0$ will be solved by induction with respect to i. Let us subdivide the i-sphere C^i into a pair of cellular i-chains A and B meeting in an $(i-1)$-sphere C^{i-1}. Then by Theorem T^i, every closed chain L^k of $H^n - C^i$ must bound two open chains L_A^{k+1} and L_B^{k+1} in $H^n - A$ and $H^n - B$, respectively. Therefore, by Corollary W^i, L^k must also bound in $H^n - C^i$ unless the closed chain $L_A^{k+1} + L_B^{k+1}$ fails to bound in $H^n - C^{i-1}$. But, by Theorem X^{i-1}, which we have a right to assume in the induction, this can only occur if $L_A^{k+1} + L_B^{k+1}$ is the $(n-i)$-chain linking C^{i-1}, in which case $k = n - i - 1$. Consequently,

$$R^s = \bar{R}^{n-s-1} = 1 \qquad (s \neq i).$$

On the other hand, the chain L^{n-i} of $H^n - C^{i-1}$ which does link C^{i-1} necessarily meets both A and B in mutually exclusive closed sets of points, for if it failed to meet A, for example, it would bound in $H^n - A$, by Theorem T^i,

and hence *a fortiori* in $H^n - C^{i-1}$, contrary to hypothesis. The chain L^{n-i} may thus be written as the sum of two open chains (cf. Lemma U^i),

$$L^{n-i} = L_A^{n-i} + L_B^{n-i} \tag{6}$$

lying in $H^n - A$ and $H^n - B$, respectively, and having a common boundary L^{n-i-1} in $H^n - C^i$. The chain L^{n-i-1} is then the required chain linking C^i. For, otherwise, there would be an open chain $\overline{L}^{n-i}$, such that

$$\overline{L}^{n-i} \equiv L^{n-i-1} \qquad (\text{mod } 2, H^n - C^i),$$

and, consequently, one or the other of the closed chains $\overline{L}^{n-i} + L_A^{n-i}$ or $\overline{L}^{n-i} + L_B^{n-i}$ would have to link C^{i-1}, since their sum $L_A^{n-i} + L_B^{n-i}$ would. But suppose, for example, that $\overline{L}^{n-i} + L_A^{n-i}$ linked C^{i-1}. Then, $\overline{L}^{n-i} + L_A^{n-i}$ would have to meet A, which would be impossible since, from their definitions, neither of its parts $\overline{L}^{n-i}$ nor L_A^{n-i} could. A similar contradiction would arise if we assumed that $\overline{L}^{n-i} + L_B^{n-i}$ linked C^{i-1}.

Finally, L^{n-i-1} is the only independent chain linking C^i. For, if M^{n-i-1} denote any $(n - i - 1)$-chain of $H^n - C^i$ linking C^i, there is associated with M^{n-i-1} a closed chain $M_A^{n-i} + M_B^{n-i}$ linking C^i, defined after the manner of the chain $L_A^{n-i} + L_B^{n-i}$ in (6) associated with L^{n-i-1}. Consequently, with the chain $L^{n-i-1} + M^{n-i-1}$, there is associated the chain

$$(L_A^{n-i} + M_A^{n-i}) + (L_B^{n-i} + M_B^{n-i})$$

which cannot link C^{i-1}. Therefore, by Corollary W^i, $L^{n-i-1} + M^{n-i-1}$ bounds in $H^n - C^i$, and M^{n-i-1} is dependent on L^{n-i-1}. Thus, finally

$$R^i = \overline{R}^{n-i-1} = 2.$$

The chain of $H^n - C^i$ which links C^i may evidently be chosen to be irreducible, for if it consisted of several irreducible parts, one at least of these parts would have to link C^i.

11. This brings us to the central theorem:

THEOREM Y. *Let C be any chain immersed in an n-sphere H^n. Then, between the invariants R^i of C and the invariants $\overline{R}^i$ of $H^n - C$ there exists the following duality relation*:

$$R^i = \overline{R}^{n-i-1} \qquad (0 \leqq i \leqq n-1).$$

To lay the foundations for a proof by induction, let us first examine the trivial case where the chain C consists of 0-cells only. Obviously, a closed chain of $H^n - C$ of dimensionality less than $n-1$ bounds as often as we please in

$H^n - C$, so that $R^{n-i-1} = R^i = 1$ $(i>0)$. To determine the remaining connectivity number $\overline{R}^{n-1}$, let us make a subdivision of H^n such that each point A_s^0 of C appears as an interior point of some cellular n-chain M_s^n of the subdivision and such that no two of the points A_s^0 belong to the same cellular n-chain M^n. The boundary of each cellular n-chain M_s^n will be a closed $(n-1)$-chain which we shall denote by L_s^{n-1}:

$$(7) \qquad M_s^n \equiv L_s^{n-1} \qquad (\text{mod } 2, H^n).$$

Now, every closed $(n-1)$-chain L^{n-1} of $H^n - C$ is homologous to some combination of the chains L_s^{n-1}; for the chain L^{n-1} surely bounds in H^n,

$$(8) \qquad M^n \equiv L^{n-1} \qquad (\text{mod } 2, H^n),$$

and if the bounded chain M^n contains points of C, we have merely to add to (8) such relations of the set (7) as correspond to the points in question to obtain an open n-chain free from points of C and bounded by L^{n-1} together with some linear combination of the chains L_s^{n-1}.

Finally, there is one and only one homology between the chains L_s^{n-1}. For by §7, any linear combination of the chains L_s^{n-1} bounds exactly two n-chains in H^n, one of which must be free of points of C if the combination is to be bound in $H^n - C$. Evidently, this can only occur if the linear combination includes all the chains L_s^{n-1}. Thus, if there be α points to C, we have

$$\overline{R}^{n-1} - 1 = \alpha - 1 = R^0 - 1,$$

by §6, which establishes the equality of $\overline{R}^{n-1}$ and R^0.

As usual, we treat by induction the case where the chain C contains at least one cell of dimensionality greater than zero.

Let B be the chain obtained by leaving off an i-cell of C of the highest dimensionality possible, and let A be the cellular i-chain determined by this i-cell. We shall assume that the theorem holds for B and shall prove that when A is restored, every change of connectivity on B is balanced by a dual change in the residual space so that the theorem continues to hold for C.

Evidently, such changes of connectivity as are produced on B by the addition of A are caused by the appearance of new independent i-chains containing A or the disappearance of independent $(i-1)$-chains by bounding open i-chains containing A. We denote the boundary of A by C^{i-1} and distinguish two cases according as

(I) The chain C^{i-1} does not bound on B;

(II) The chain C^{i-1} bounds some open i-chain A^i of B.

In the first case, no new closed i-chain can be created, since such a chain would be of the form $A + A^i$ which would imply

$$A^i \equiv C^{i-1} \qquad \text{(mod 2, } B\text{)},$$

contrary to hypothesis. On the other hand, we have

$$A \equiv C^{i-1} \qquad \text{(mod 2, } C\text{)},$$

so that the independent chain C^{i-1} is lost. No other independent chain is lost, for a second relation

$$A + D^i \equiv D^{i-1} \qquad \text{(mod 2, } C\text{)}$$

would imply

$$D^i \equiv C^{i-1} + D^{i-1} \qquad \text{(mod 2, } B\text{)}$$

showing that D^{i-1} was dependent on C^{i-1}.

The second case is treated with equal facility. Without going into details, we find that a single independent i-chain $A + A^i$ is gained and that no independent $(i-1)$-chain is lost.

To calculate the compensating changes of connectivity produced in the residual space, we define a closed $(n-i-1)$-chain of the residual space which will be said to be *dual* to the chain A. Let an irreducible $(n-i)$-chain L^{n-i} of $H^n - C^{i-1}$ be chosen linking the boundary C^{i-1} of A and therefore meeting A in a closed set of points. Whenever we can, we shall choose the chain L^{n-i} in such a way that it contains at least one point not of A. It will then be possible, by the process already so frequently employed, to break the chain L^{n-i} up into a pair of open chains bounded by a closed chain L^{n-i-1} of $H^n - C^i$ and such that one of the open chains which we shall call M^{n-i} contains all the points of intersection of L^{n-i} with A but no point of B:

$$(9) \qquad M^{n-i} \equiv L^{n-i-1} \qquad \text{(mod 2, } H^n - \mathrm{B}\text{)}.$$

The chain L^{n-i-1} will be said to be dual to A. It evidently links any i-sphere contained in C and containing A, for it has been obtained by exactly the construction given in the proof of Theorem X^i for finding the chain linking such an i-sphere. Moreover, as we note for future reference, if ϵ be any positive constant, the chain M^{n-i} bounded by L^{n-i-1} may evidently be so chosen that each of its points is within a distance ϵ of some point of intersection of M^{n-i} with A.

We derived the dual L^{n-i-1} of A on the assumption that a chain L^{n-i} linking C^{i-1} could be found which contained a point not of A. Now, as a matter of fact, it is easy to prove that any chain linking C^{i-1} must contain such a point. Rather than digress to prove this, however, let us merely say that if no chain

L^{n-i} contains a point not of A, then $M^{n-i} = L^{n-i}$ and the boundary of M^{n-i} is the "null" $(n-i-1)$-chain.

We now prove two lemmas.

(α) *If a closed chain L^{k+1} of $H^n - B$ does not link the boundary C^{i-1} of A, there is always some chain of $H^n - C$ (that is to say, some chain not meeting A) which is homologous to L^{k+1} in $H^n - B$. If the chain L^{k+1} does link C^{i-1}, however, there is no chain of $H^n - C$ homologous to L^{k+1}.*

For if L^{k+1} links C^{i-1} it must cut the cellular i-chain A, otherwise it would bound in $H^{n+1} - A$, by Theorem T^i, and therefore *a fortiori* in $H^{n+1} - C^{i-1}$. The same is true of all chains homologous to L^{k+1}.

On the other hand, if L^{k+1} does not link C^{i-1}, there exists an open chain M^{k+2} such that

$$M^{k+2} \equiv L^{k+1} \qquad (\text{mod } 2, H^n - C^{i-1}).$$

We may again use the figure going with Lemma U^i to represent the situation schematically, provided we let $L^{k+1}_A + L^{k+1}_B$ represent the chain L^{k+1} and imagine that L^{k+1}_A is nearer to the eye than B and does not intersect B, though M^{k+2} may. Now M^{k+2} meets A and B, if at all, in mutually exclusive closed sets of points, therefore we may repeat the argument of Lemma U^i and find a chain $L^{k+1} + \bar{L}^{k+1}$ which does not meet A (nor B in this case). But, with the same notation as before,

$$\bar{M}^{k+2} \equiv \bar{L}^{k+1} \sim 0 \qquad (\text{mod } 2, H^n - B).$$

Therefore,

$$L^{k+1} + \bar{L}^{k+1} \sim L^{k+1},$$

showing that there remains in $H^n - C$ a representative of the family of chains homologous to L^{k+1} in $H^n - B$.

Thus, in view of Theorem X^i, the addition of A to B cannot reduce the number of independent non-bounding chains of the residual space aside from those of dimensionality $n - i$. Moreover, not more than one independent $(n - i)$-chain can be destroyed, since the sum of two chains linking C^{i-1} cannot link C^{i-1}.

(β) *Let L^k be a closed chain of $H^n - C$ which bounds in $H^n - B$;*

$$L^{k+1}_B \equiv L^k \qquad (\text{mod } 2, H^n - B)$$

and, necessarily, of course, in $H^n - A$

$$L^{k+1}_A \equiv L^k \qquad (\text{mod } 2, H^n - A), \tag{10}$$

by Theorem T^i. Then, L^k fails to bound in $H^n - C$ if and only if the closed chain

$$L_A^{k+1} + L_B^{k+1} \equiv 0 \qquad (\text{mod } 2,\ H^n - C^{i-1}) \tag{11}$$

links C^{i-1} for every posible choice of the chain L_B^{k+1}.

For, as we saw in (α), $L^{k+1} = L_A^{k+1} + L_B^{k+1}$ must cut A if it links C^{i-1}. Consequently, L_B^{k+1} must cut A, since L_A^{k+1} does not (see figure). But if this occurs for every choice of L_B^{k+1}, L cannot bound in $H^n - C$.

On the other hand, if L_B^{k+1} can be so chosen that $L_A^{k+1} + L_B^{k+1}$ does not link C^{i-1}, then, by Corollary W^i, L^k must bound in $H^n - C$. As an immediate consequence of (β), no new independent non-bounding closed chains of dimensionalities other than $n - i - 1$ can be created in the residual space when A is added to B.

Let us now combine the two results obtained.

(I′) If the space $H^n - B$ contains an $(n - i)$-chain L linking C^{i-1}, no new independent $(n - i - 1)$-chain can be created in the residual space. For, in place of relation (10) in Lemma (β), we may equally well write

$$L + L_A^{k+1} \equiv L^k \qquad (\text{mod } 2,\ H^n - A), \tag{10′}$$

(where $k = n - i - 1$), and so obtain in place of (11) the expression

$$(L + L_A^{k+1}) + L_B^{k+1} \equiv 0. \tag{11′}$$

But not both of the chains (11) and (11′) can link C^{i-1}; therefore L^k must bound in $H^n - C$, by Corollary W^i. The net result of adding A to B is therefore to diminish by unity the number $\bar{R}^{n-i}$ but to leave invariant the remaining connectivity numbers of the residual space.

(II′) If the space $H^n - B$ contains no $(n - i)$-chain linking C^{i-1}, none of the connectivity numbers $\bar{R}^s$ can be diminished. However, in this case, a chain of $H^n - C^{i-1}$ linking C^{i-1} must meet both A and B. Consequently by a literal transcription of the proof of Theorem X^i, we find that a single new independent $(n - i - 1)$-chain is created which is nothing more than the dual of A. The number $\bar{R}^{n-i-1}$ is therefore increased by unity. It also follows that every independent non-bounding $(n - i - 1)$-chain of $H^n - C$ is homologous in that region to some linear combination of the duals of the i-cells of C.

Now, it will be observed that the changes of connectivity (I′) and (II′) of the residual space are exactly the ones wanted to compensate for the changes of connectivity (I) and (II), respectively, of the immersed figure. Furthermore, whether or not similarly numbered changes occur together when A is added to B, the differences $R^i - R^{i-1}$ and $\bar{R}^{n-i-1} - \bar{R}^{n-i}$ both increase by unity in every case and therefore remain equal to one another. Thus to complete the proof we

have only to show that the number R^i of C is equal to the number $\bar{R}^{n-i-1}$ of $H^n - C$. This we proceed to do.

Let C' be the chain obtained by eliminating all the cells of C of highest dimensionality i. Then, to each cellular $(i-1)$-chain A' of C' may be found a dual $(n-i)$-chain L^{n-i} of $H^n - C'$ so close to A' that it meets only such i-cells of C as contain A' on their boundaries. The latter cells it must meet, however, since it links their boundaries. We may therefore break up the chain L^{n-i} into a set of open i-chains each containing the points of intersection of L^{n-i} with one alone of the i-cells of C and each bounded by the dual of the chain determined by that i-cell. Thus, if L_p^{n-i-1} denote the dual of A_p^i, we shall have a set of homologies

$$\bar{E}_q \qquad \sum_p \epsilon_{qp} L_p^{n-i-1} \sim 0 \qquad (\text{mod } 2,\ H^n - C)$$

where the coefficients ϵ_{qp} have precisely the same significance as in the relations E_q of (§ 6). Furthermore, every homology among the duals L_p^{n-i-1} of chains A_p^i is expressible as a linear combination of the fundamental homologies $\bar{E}_q$. For suppose

$$(12) \qquad M^{n-i} \equiv \sum_u \alpha_u L_u^{n-i-1} \sim 0 \qquad (\text{mod } 2,\ H^n - C).$$

Then, by combining with (9), we obtain the closed chain

$$M^{n-i} + \sum_u \alpha_u M_u^{n-i} \equiv 0 \qquad (\text{mod } 2,\ H^n - C').$$

But this chain is expressible linearly in terms of the duals of the cellular $(i-1)$-chains of C'so that the homology (12) is expressible in terms of the homologies $\bar{E}_q$. We therefore have

$$R^i - 1 = \alpha - \rho = \bar{R}^{n-i-1} - 1$$

(cf. § 6), which establishes the equality of R^i and $\bar{R}^{n-i-1}$.

12. In closing we give a few corollaries of the fundamental theorem.

Jordan-Brouwer Theorem. *If M^{n-1} be an $(n-1)$ dimensional manifold* (§ 6) *immersed in an n-sphere H^n, the residual domain $H^n - M^{n-1}$ consists of exactly two connected regions.* For $\bar{R}^0 = \bar{R}^{n-1} = 2$.

Let us distinguish between these regions by arbitrarily calling one the *interior* and the other the *exterior* of M^{n-1}. Then,

If the manifold M^{n-1} be homeomorphic with an $(n-1)$-sphere, the connectivity numbers of both interior and exterior are unity. For $\bar{R}^i = R^{n-i-1} = 1$ $(i > 0)$.

ACCESSIBILITY THEOREM. *In every neighborhood of every point of M^{n-1}, there is a point of M^{n-1} which is accessible from any point P of $H^n - M^{n-1}$ by a broken line of geodesic arcs made up of points of $H^n - M^{n-1}$.*

For consider any subdivision of M^{n-1} into cells. Then, it is always possible to join the point P to a point Q on the other side of M^{n-1} by a broken line which meets M^{n-1} in one $(n-1)$-cell of the subdivision only. Because, if that one $(n-1)$-cell were omitted, we should have $R^{n-1} = \bar{R}^0 = 1$ and the space residual to M^{n-1} would become connected. But this $(n-1)$-cell may be chosen in an arbitrary neighborhood of an arbitrary point of M^{n-1}, which proves the theorem and also the following corollary:

There are interior and exterior points in every neighborhood of every point of M^{n-1}; for example, points of the broken line PQ.

By the same device of omitting an arbitrarily small $(n-1)$-cell, the theorem on the invariance of dimensionality may be proved.

Let C^k be any cellular k-chain ($k \leqq n-1$) immersed in an n-sphere H^n. Then, there are points of $H^n - C^k$ in every neighborhood of every point of C^k; namely, points of a 1-chain linking the boundary of the omitted $(n-1)$-cell.

The theorem on the invariance of regionality is, of course, an immediate consequence of the separation theorem:

Let C^n be a cellular n-chain immersed in an n-sphere H^n. Then no interior point of C^n is a limit point of points of $H^n - C^n$. For the boundary of C^n is an $(n-1)$-dimensional manifold separating H^n into $H^n - C^n$ and the interior of C^n.

Finally, we note that Theorem Y establishes the purely topological character of the invariants R^i and $\bar{R}^{n-i-1}$. For $\bar{R}^{n-i-1}$ does not depend at all upon the particular cellular structure of C, but only on the set of points determined by C; therefore, the same must be true of R^i. Conversely, R^i is not affected by the choice of the metric on H^n; therefore, neither is $\bar{R}^{n-i-1}$.

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OSCILLATION THEOREMS IN THE COMPLEX DOMAIN*

BY

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I. Introduction

The aim of the present paper is to throw some light on the question of the distribution in the complex domain of the zeros of functions satisfying linear homogeneous differential equations of the second order. The real zeros of such functions are well known from the works of numerous mathematicians from Sturm and Liouville down to living writers, but our knowledge of the complex zeros is very deficient. It is only in special cases that progress has been made.

The number of complex zeros of a hypergeometric function in the case of real parameters has been determined by Hurwitz, Van Vleck and Schafheitlin. Hurwitz has also investigated Bessel functions. The same field has been covered by Macdonald, Porter and Schafheitlin.

Hurwitz used in his paper on Bessel functions‡ certain integral equalities analogous to those frequently used for establishing the reality of the characteristic values in a boundary problem.§ In his thesis|| the present writer used similar equalities for the study of the zeros of Legendre functions.

The present paper contains a systematic study of integral equalities, called *Green's transforms*, which are adjoined to linear differential equations of the second order. It is shown that these equalities give information concerning the distribution of the zeros of a function satisfying such an equation. From the knowledge that a particular solution of the equation in question vanishes at a point in the complex plane, is real on an interval, or similar information, we are able to assign certain regions of the plane, containing the point or the interval where this particular solution cannot vanish. In such a fashion we can estab-

* Presented to the Society, October 28, 1922.

† The author wishes to express his most cordial thanks to Professor G. D. Birkhoff for constant encouragement and for many valuable suggestions received during the work on the present paper.

‡ Mathematische Annalen, vol. 33 (1889), pp. 246–266.

§ Cf. for instance, Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, pp. 73–76.

|| *Some problems concerning spherical harmonics*, Arkiv för Matematik, Astronomi och Fysik, vol. 13 (1918), No. 17.

lish the existence of various *zero-free regions*. It is often possible by simple means to make these zero-free domains cover the greater part of the plane, so that the zeros of the solution are distributed over a comparatively small region.

The investigation is completed by a study of the asymptotic distribution of the zeros by means of a singular integral equation of Volterra's type. This latter method often gives information on the general disposition of the zeros and forms a valuable complement to the method of assigning zero-free regions.

In the greater part of the paper we treat a special differential equation, $w'' - w/z = 0$, for purposes of illustration. The solutions of this equation are expressible in terms of Bessel functions of the first order.

The author has applied his methods with success to the study of several special differential equations such as, for instance, Bessel's, Legendre's,* Mathieu's and Weber's equations. The results will be published in later papers.

2. Green's transform

2.1. Transformation formulas. We can always assume that our differential equation is written in self-adjoint form

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0. \tag{2.11}$$

The functions $G(z)$ and $K(z)$ we suppose to be analytic in a region T of the z-plane, where, further, $K(z)$ does not vanish. We can replace the equation (2.11) by the system

$$\begin{aligned} \frac{dw_1}{dz} &= \frac{1}{K(z)}\,w_2; \\ \frac{dw_2}{dz} &= -\,G(z)w_1; \end{aligned} \tag{2.12}$$

where we have put

$$\begin{aligned} w_1 &= w; \\ w_2 &= K(z)\,\frac{dw}{dz}. \end{aligned} \tag{2.121}$$

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* *On the zeros of Legendre functions*, Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1922), No. 22.

If we further replace the first equation in (2.12) by its conjugate, the system becomes

$$
\begin{aligned}
d\bar{w}_1 &= \frac{1}{\overline{K(z)}}\,\bar{w}_2 d\bar{z}, \\
dw_2 &= -G(z)w_1 dz.
\end{aligned} \tag{2.122}
$$

Multiply the first equation by w_2, the second by $\bar{w}_1$, add, and integrate the result between the limits z_1 and z_2. Thus we get

$$
\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \frac{d\bar{z}}{\overline{K(z)}} + \int_{z_1}^{z_2} |w_1|^2 G(z) dz = 0. \tag{2.13}
$$

Here we have assumed z_1 and z_2 to be points in T and also the path of integration to be entirely in T. This expression (2.13), which is of fundamental importance for our coming work, we shall call the *Green's transform* of the differential equation on account of its similarity to a well known formula of Green for real variables.

If we write

$$
\begin{aligned}
\frac{dz}{K(z)} &= d\mathbf{K}_1 + i\,d\mathbf{K}_2 = d\mathbf{K}, \\
G(z)dz &= d\mathbf{\Gamma}_1 + i\,d\mathbf{\Gamma}_2 = d\mathbf{\Gamma},
\end{aligned} \tag{2.14}
$$

we have

$$
\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 d\overline{\mathbf{K}} + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma} = 0, \tag{2.15}
$$

or, if the real and the imaginary parts be taken,

$$
\mathfrak{R}\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_1 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_1 = 0, \tag{2.151}
$$

$$
\mathfrak{Im}\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_2 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_2 = 0. \tag{2.152}
$$

2.2. **Change of variables.** Let us introduce a new independent variable in system (2.12) by putting

$$
dz = f(Z)\,dZ, \tag{2.21}
$$

whereby the system is transformed into

$$
\begin{aligned}
dw_1 &= \frac{1}{k(Z)} w_2 dZ, \\
dw_2 &= -\, g(Z) w_1 dZ,
\end{aligned} \tag{2.22}
$$

with

$$k(Z) = \frac{K(z)}{f(Z)}, \quad g(Z) = G(z)f(Z).$$

The Green's transform goes over into

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \, d\bar{\kappa} + \int_{Z_1}^{Z_2} |w_1|^2 \, d\gamma = 0, \tag{2.23}$$

where

$$
\begin{aligned}
\frac{dZ}{k(Z)} &= d\kappa, \\
g(Z)dZ &= d\gamma.
\end{aligned} \tag{2.231}
$$

Thus we infer that *the Green's transform is invariant under a transformation of the independent variable.*

By special choice of Z we can obtain simpler forms of the transform. Our first choice will be

$$Z = \mathbf{K}(z). \tag{2.24}$$

If we put

$$J(Z) = G(z)K(z),$$

we obtain

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \, d\bar{Z} + \int_{Z_1}^{Z_2} |w_1|^2 J(Z) dZ = 0, \tag{2.241}$$

to which formula corresponds a differential equation

$$\frac{d^2 w}{dZ^2} + J(Z) w = 0. \tag{2.242}$$

This special type of differential equation is the one with which we shall be concerned mostly in this paper.

We can of course also choose

$$Z = \mathbf{\Gamma}(z),^* \tag{2.25}$$

* This function has of course nothing to do with the classical Γ-function.

which yields

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \frac{d\bar{Z}}{Q(Z)} + \int_{Z_1}^{Z_2} |w_1|^2 \, dZ = 0, \tag{2.251}$$

with

$$Q(Z) = G(z)K(z),$$

corresponding to the differential equation

$$\frac{d}{dZ}\left[Q(Z)\frac{dw}{dZ}\right] + w = 0. \tag{2.252}$$

Finally we can get a more symmetric form by putting

$$dZ = \sqrt{\frac{G(z)}{K(z)}}\, dz, \tag{2.26}$$

which leads to

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \frac{d\bar{Z}}{S(Z)} + \int_{Z_1}^{Z_2} |w_1|^2 S(Z) dZ = 0, \tag{2.261}$$

with

$$S(Z) = \sqrt{G(z)K(z)}.$$

The differential equation takes the following form:

$$\frac{d}{dZ}\left[S(Z)\frac{dw}{dZ}\right] + S(Z)w = 0. \tag{2.262}$$

This transformation we shall use in the last section of the paper for the study of the solutions in the neighborhood of an irregular singular point.

Up to this point we have changed only the independent variable. We can of course also make a simultaneous transformation of both independent and dependent variables

$$\begin{aligned} z &= f(Z), \\ w &= g(Z)W. \end{aligned} \tag{2.27}$$

Starting from an equation of the form (2.242), for instance, we can find infinitely many transformations of type (2.27), which furthermore preserve the form of the differential equation. For each set of such variables W, Z we have a Green's transform of type (2.241) of our equation.

The simplest possible of all such transformations is

$$\begin{aligned} w &= W, \\ z &= aZ + b, \end{aligned} \tag{2.28}$$

which carries $J(z)$ into $a^2 J(aZ + b)$. We shall return to this transformation in §3.8.

It is evident that w_1 and w_2 play the same rôle in the system (2.12). From this remark it follows that we have also formulas like

$$\left[w_1\bar{w}_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \frac{dz}{K(z)} + \int_{z_1}^{z_2} |w_1|^2 G(z)\, \overline{dz} = 0. \tag{2.29}$$

This is of course nothing but the conjugate of the expression in formula (2.13). We shall find later on that in all theorems concerning distribution of zeros, w_1 and w_2 enter symmetrically.

2.3. **Two nets of curves.** We can evidently simplify the formulas (2.151) and (2.152) by choosing the path of integration so that one of the following relations is fulfilled, namely

$$\begin{aligned} d\mathbf{K}_1 &= 0; \qquad d\mathbf{K}_2 = 0; \\ d\mathbf{\Gamma}_1 &= 0; \qquad d\mathbf{\Gamma}_2 = 0. \end{aligned} \tag{2.31}$$

These paths constitute four families of curves which together form two independent nets of orthogonal trajectories.

In the case of an equation of type (2.242) the $\mathbf{K}$-net is simply the net of lines parallel to the axes. The same lines serve as a $\mathbf{\Gamma}$-net of an equation of type (2.252).

On account of the skew symmetry between $G(z)$ and $K(z)$ we need consider only one of the two nets. Our results will hold, mutatis mutandis, for the other net. We choose the $\mathbf{\Gamma}$-net, as the $\mathbf{K}$-net is trivial in the most important of all special cases, namely that in which $K(z) = 1$.

Through every point a of the region T in the z-plane where $G(z)$ is regular and furthermore $G(a) \neq 0$, passes one and only one curve of each of the families $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, which curves we denote by $\mathbf{\Gamma}_1 a$ and $\mathbf{\Gamma}_2 a$, respectively. These two curves are, of course, orthogonal to each other, with slopes equal to $g_1(a)/g_2(a)$ and $-g_2(a)/g_1(a)$, respectively, if we put

$$G(z) = g_1(z) + ig_2(z). \tag{2.32}$$

Hence the $\mathbf{\Gamma}_1$-family has horizontal tangents along the curves $g_1(z) = 0$ and vertical ones along $g_2(z) = 0$. For the $\mathbf{\Gamma}_2$-curves the state of affairs is reversed.

We shall now proceed to consider the behavior of the $\mathbf{\Gamma}$-curves in the neighborhood of an exceptional point of $G(z)$ where this function either vanishes or becomes infinite.* We restrict ourselves to the case where $G(z)$ is analytic in T except for poles.

* Cf. for the theory of curves of this kind F. Lucas, *Géométrie des polynômes*, Journal de l'École Polytechnique, vol. 28 (1879), and F. Klein, *Ueber Riemann's Theorie der Algebraischen Funktionen und ihrer Integrale*, Leipzig, 1882.

First, let $z = a$ be a k-fold zero of $G(z)$. Then

$$\int_a^z G(z)dz = a_k \frac{(z-a)^{k+1}}{k+1} + \dots .$$

If $z - a = re^{i\theta}$ and $a_k = \rho e^{i\varphi}$ we have

$$\mathfrak{R}\left\{\int_a^z G(z)dz\right\} = \frac{\rho}{k+1} r^{k+1} \cos\Big[(k+1)\theta + \varphi\Big] + \cdots;$$
$$\mathfrak{Im}\left\{\int_a^z G(z)dz\right\} = \frac{\rho}{k+1} r^{k+1} \sin\Big[(k+1)\theta + \varphi\Big] + \cdots.$$

Hence $(k+1)$ branches of $\mathbf{\Gamma}_1 a$ pass through $z = a$ and the same number of branches of $\mathbf{\Gamma}_2 a$ intersect them there in such a manner that the tangents at a of the one curve make equal angles with each other and bisect the angles formed by the tangents of the other curve.

If $z=a$ is a pole of order $k > 1$ of $G(z)$, we have

$$\int^z G(z)dz = C - \frac{a_k}{k-1} \frac{1}{(z-a)^{k-1}} + \cdots.$$

With the same notation as above we obtain

$$\mathfrak{R}\left[\int^z G(z)dz\right] = \gamma - \frac{\rho}{k-1} r^{1-k} \cos\Big[(k-1)\theta - \varphi\Big] + \cdots;$$
$$\mathfrak{Im}\left[\int^z G(z)dz\right] = \delta + \frac{\rho}{k-1} r^{1-k} \sin\Big[(k-1)\theta - \varphi\Big] + \cdots,$$

where $\gamma + i\delta = C$. Let us assume for the sake of simplicity that no logarithmic term appears. Then we can conclude that every curve of the two families which passes in the neighborhood of $z = a$ actually passes through this point. The $\mathbf{\Gamma}_1$-curves are tangent to the lines

$$\arg(z-a) = \left[\varphi + (2\nu+1)\frac{\pi}{2}\right]/(k-1) \quad (\nu = 0, 1, \dots, k-2).$$

Replacing $2\nu + 1$ by 2ν we get the tangents to the $\mathbf{\Gamma}_2$-curves.

In the case where $z = a$ is a simple pole of $G(z)$ we have

$$\int^z G(z)dz = C + a_1 \log(z-a) + \cdots$$

and

$$\mathfrak{R}\left[\int^z G(z)dz\right] = \gamma + \alpha_1 \log r - \beta_1 \theta + \cdots ;$$

$$\mathfrak{Im}\left[\int^z G(z)dz\right] = \delta + \beta_1 \log r + \alpha_1 \theta + \cdots ,$$

where $\gamma + \delta i = C$, and $\alpha_1 + \beta_1 i = a_1$.

If neither α_1 nor β_1 is zero all curves of the net in the vicinity of $z = a$ admit this point as an asymptotic point which they approach in the manner of a logarithmic spiral. When $\beta_1 = 0$ all curves of $\mathbf{\Gamma}_1$ sufficiently near to a are ovals around this point which are approximately circular for great values of γ. In the first approximation, the $\mathbf{\Gamma}_2$-family behaves like a pencil of lines through a. If $\alpha_1 = 0$ the rôles of the two families are interchanged.

When carrying over the results of the discussion of the descriptive properties of the $\mathbf{\Gamma}$-net to the $\mathbf{K}$-net, we have to observe that the rôles of zeros and of poles are interchanged; a zero of $K(z)$ being a pole or logarithmic point of $\mathbf{K}(z)$ and so on.

2.4. $G(z)$ **a polynomial.** Suppose $G(z)$ is a polynomial of degree n

$$G(z) = A_0(z - a_1)^{\nu_1} (z - a_2)^{\nu_2} \dots (z - a_\mu)^{\nu_\mu} \qquad (\nu_1 + \nu_2 + \dots + \nu_\mu = n).$$

A non-specialized curve belonging either to $\mathbf{\Gamma}_1$ or $\mathbf{\Gamma}_2$ has no double points at all in the projective plane. Such a curve does not pass through any of the points $a_1, a_2, \cdots, a_\mu$. There is, however, one curve of each family which passes through the point a_k and has a $(\nu_k + 1)$-tuple point there. Thus there are at most μ singular curves in each family.

The intersections with the line at infinity in the projective plane are all distinct for one and the same curve but the same for all curves of the family. The asymptotes are all real and distinct; they all intersect in a fixed point, the center of gravity of the roots $(\nu_1 a_1 + \nu_2 a_2 + \cdots + \nu_\mu a_\mu)/n$, where they furthermore form equal angles with each other. If $\arg A_0 = \varphi_0$ the asymptotic directions of a $\mathbf{\Gamma}_1 a$ are given by

$$\arg z_k = \frac{2k+1}{n+1}\frac{\pi}{2} - \frac{\varphi_0}{n+1} \qquad (k = 0, 1, \dots, n).$$

The asymptotes of the $\mathbf{\Gamma}_2$-curves pass through the same point and bisect the angles between the asymptotes of the $\mathbf{\Gamma}_1$-family. We note that the asymptotes of the curves $g_1(z) = 0$ and $g_2(z) = 0$ also intersect in this point.

A $\mathbf{\Gamma}$-curve can never begin or end in a finite point, nor can it be closed. This implies that every point in the plane can be reached along a suitably chosen

path from infinity without crossing the curve in question. This is a consequence of the fact that $\Gamma_1(z)$ and $\Gamma_2(z)$, being harmonic functions throughout the finite plane, can not have maxima or minima.

3. Determination of zero-free regions

3.1. **The real axis.** We shall apply the results of the preceding section to the problem of locating the zeros of a given solution of a linear differential equation of the second order. We start with the real zeros and restrict ourselves to the case when $K(z) = 1$. From formulas (2.151) and (2.152) we obtain

$$\Re\Big[w_1 w_2\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} |w_2|^2 dx + \int_{x_1}^{x_2} g_1(x)|w_1|^2 dx = 0; \tag{3.11}$$

$$\Im\mathrm{m}\Big[w_1 w_2\Big]_{x_1}^{x_2} + \quad * \quad + \int_{x_1}^{x_2} g_2(x)|w_1|^2 dx = 0. \tag{3.12}$$

Now take an arbitrary solution w of the equation (2.242) or a pair of solutions (w_1, w_2) of the corresponding system. Can this solution w, or, more generally, can the product $w_1 w_2$ vanish at two points x_1 and x_2 of a certain interval (a, b) of the real axis? Suppose $g_2(z)$ keeps a constant sign in (a, b). Then by (3.12) $w_1 w_2$ can vanish once at most in the interval in question, and similarly if $g_1(z) \leqq 0$ in (a, b). We have assumed of course that $G(z) = J(z)$ is analytic in the interval. Thus

Theorem 3.1. *In an interval of the real axis throughout which $J(z)$ is analytic and furthermore $\Re[J(z)] \leqq 0$ or $\Im\mathrm{m}[J(z)]$ keeps a constant sign (perhaps vanishing at discrete points) there can be at most one zero of $w\,dw/dz$ where w is an arbitrary solution of $w'' + J(z)w = 0$.*

Thus if w is to oscillate on the axis of reals, supposed free from singular points, we must have $g_1(z) \geqq 0$ and $g_2(z)$ must change sign or vanish identically.

In case there are singular points on the real axis we may be able to get additional information concerning the zeros. Suppose for instance that $z = a$ is a regular singular point of the equation with exponents λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$, $\Re(\lambda_1) > \Re(\lambda_2)$. Then $J(z)$ has a pole of at most the second order at $z = a$. If we assume $\Re(\lambda_1) > 1/2$ the expressions in (3.11) and (3.12) will remain finite when we let x_1 or x_2 converge towards a, provided w_1 is the solution corresponding to the exponent λ_1; and $w_1 w_2 \longrightarrow 0$ when $x \longrightarrow a$. Consequently this particular solution will not vanish in an interval (a, b), a being the singular point, where $g_1(z)$ and $g_2(z)$ are subject to the conditions stated in the theorem above. This remark, of course, also applies to more general types of singular points.

It is obvious that the results of this section can be generalized to equations of the type (2.11).

3.2. **Dynamical interpretation.** Theorem 3.1 admits of direct dynamical interpretation.

If we put $w = u + iv$ in equation (2.242), assume $z = x$ to be real and separate reals and imaginaries, we obtain the following system

$$(3.21)\qquad \frac{d^2u}{dx^2} = -g_1u + g_2v; \qquad \frac{d^2v}{dx^2} = -g_1v - g_2u.$$

These are the equations for the motion of a particle of unit mass, subjected to a force whose components on the radius vector and on a line perpendicular to the same are $g_1\sqrt{u^2+v^2}$ and $-g_2\sqrt{u^2+v^2}$, respectively.

Theorem 3.1 now gives a sufficient condition that a particle starting from the origin in the u,v-plane at the time x_1 with a certain velocity shall continue to move away from the origin during a certain space of time. The condition $g_1(x) \leqq 0$ evidently means that the radial component of the force is always directed from the origin whereas the invariability of sgn $g_2(x)$ means that the rotating component has the same sense during the time interval in question. These conditions are evidently sufficient to prevent the particle from turning towards the origin.

3.3. **Linear paths.** We shall now consider an arbitrary linear segment (z_1, z_2) in the complex plane and an arbitrary solution $w(z)$ of equation (2.242) under the assumption that $J(z)$ is analytic along the segment chosen. Put

$$(3.31)\qquad z = z_1 + re^{i\theta}.$$

From (2.241) we infer

$$(3.32)\qquad \left[w\frac{dw}{dr}\right]_0^r - \int_0^r \left|\frac{dw}{dr}\right|^2 dr + e^{2i\theta}\int_0^r |w|^2 J(z)dr = 0.$$

If we put

$$w\frac{dw}{dr} = f_1(r) + if_2(r);$$

$$\cos 2\theta\, g_1(z) - \sin 2\theta\, g_2(z) = P(z,\theta); \quad \cos 2\theta\, g_2(z) + \sin 2\theta\, g_1(z) = Q(z,\theta),$$

we obtain

$$(3.33)\qquad f_1(r) - f_1(0) - \int_0^r \left|\frac{dw}{dr}\right|^2 dr + \int_0^r P(z,\theta)|w|^2 dr = 0;$$

$$(3.34)\qquad f_2(r) - f_2(0) + \int_0^r Q(z,\theta)|w|^2 dr = 0.$$

These formulas yield the following

THEOREM 3.31. *There is at most one zero of the product $w\,dw/dz$, where $w(z)$ is an arbitrary solution of $w'' + J(z)w = 0$, on a segment (z_1, z_2) in the complex plane where $J(z)$ is analytic, provided either*

(i) *$P(z, \theta) \leqq 0$ along the segement, or*

(ii) *$Q(z, \theta)$ keeps a constant sign there.*

If in addition to (i) *$f_1(0) \geqq 0$, or in addition to* (ii) *$f_2(0)$ has opposite sign to that of $Q(z, \theta)$ all along the segment, then there is no zero at all of the product $w\,dw/dz$ on (z_1, z_2).*

Let us now extend this theorem to a region T in the complex plane where $J(z)$ is analytic. Take a pencil of parallel lines l_{z_0};

$$z = z_0 + re^{i\theta}$$

Each line is characterized by the variable point z_0. Let T be simply connected and of such a shape that every line of the pencil cuts the boundary in two points at most. Then there will be two lines of l_{z_0} each of which meets the boundary in two coincident points, α and β, respectively. These two points α and β divide the boundary in two parts, one of which we shall designate by C and which shall be the locus of the points z_0. Then we have

THEOREM 3.32. *In a region T, defined as above, there is at most one zero of the product $w\,dw/dz$ on any line l_{z_0} if either*

(i) *$P(z, \theta) \leqq 0$ throughout T, or*

(ii) *$Q(z, \theta) \neq 0$ in T.*

If in addition to (i) *$\Re[e^{i\theta}\overline{w(z)}dw/dz] \geqq 0$ along C, or in addition to* (ii) *$\Im\mathrm{m}[e^{i\theta}\overline{w(z)}dw/dz] = 0$ or has opposite sign to that of $Q(z, \theta)$ in T all along C, then there is no zero of the product in T.*

The first part of the theorem does not lay any serious restrictions on the zeros of w, and the second part seems rather unmanageable. We shall however give some special cases of great interest. Suppose C is a segment of the real axis, $\theta = \pi/2$ and $w(z)$ takes on only real values on C. Then $\Re[e^{i\theta}\overline{w}dw/dz] = 0$ and we obtain

THEOREM 3.321. *If the equation* (2.242) *has a solution $w(z)$ which is real on a segment (a, b) of the real axis; if further T is a region symmetric with reference to (a,b) such that every vertical line which cuts the region cuts its boundary twice and meets (a,b) in an interior point; and if finally $\Re[J(z)] \geqq 0$ throughout T, then neither $w(z)$ nor dw/dz can have any* complex *zeros in T.*

If in the statement of the preceding theorem we replace the real axis by the axis of imaginaries, vertical lines by horizontal and finally suppose $\Re[J(z)] \leqq 0$ in T the theorem still holds, the exceptional zeros now being purely imaginary.

As a simple application we may choose the equation $w'' + w = 0$ with $w = \sin z$. Theorem 3.321 shows that there are no complex zeros of $\sin z$ or $\cos z$.

We can of course also derive a special theorem from formula (3.33), namely

THEOREM 3.322. *If the equation* (2.242) *has a solution $w(z)$, real on a segment (a,b) of the real axis where $w\, dw/dz$ keeps a constant sign, namely the same as that of $\mathfrak{Im}[J(z)]$ in a region T above the real axis, whose base is (a,b) and of which the boundary above the real axis is cut in one point at most by any vertical line; then neither w nor dw/dz can have a complex zero in T or in the symmetric region $\overline{T}$ below the real axis.*

A similar theorem can be stated for the imaginary axis.

As a simple illustration take the equation

$$\frac{d^2w}{dz^2} - \frac{1}{z}\, w = 0. \tag{3.35}$$

This equation will be used throughout the remainder of the paper for the purpose of giving concrete applications of our theorems. Thus we need some knowledge of the nature of its solutions. ∞ is an irregular singular point whereas 0 is regular singular with exponents 0 and 1. There is one solution, $E(z) = \sqrt{z}\, J_1(2i\sqrt{z})$, where $J_1(u)$ denotes the ordinary Bessel function of order 1, which is an entire function of z and vanishes at the origin; every other solution is of the form $[c_1 + c_2 \log z]E(z) + c_2H(z)$ where $H(z)$ is an entire function. $E(z)$ is real for all real values of z and has infinitely many real negative zeros; every other solution can be real on a half-axis at most and if it is real on the negative half of the real axis it must oscillate there. A solution which is real for $x > 0$ can have at most one positive zero or extremum. In general w^2 will increase beyond all limit when $x \longrightarrow +\infty$. In fact we have

$$\lim_{x\to+\infty} \frac{w'(x)\sqrt{x}}{w(x)} = +1, \tag{3.351}$$

save for one exceptional integral which yields -1 in the limit.*

As $\mathfrak{R}(-1/z) > 0$ when $R(z) = x < 0$ we infer by theorem 3.321 that no integral of the equation which is real for real negative values can have any complex zeros in the left half-plane. Further $\mathfrak{Im}(-1/z) > 0$ when $x > 0$. Take a solution which is real for real positive values of z and let x_0 be the positive zero of $w(z)\, dw/dz$ if there is any; otherwise positive but arbitrary. Then by theorem 3.322 we infer that such a solution can not vanish in the half-plane $\mathfrak{R}(z) \geqq x_0$.

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* Cf. A. Wiman, *Über die reellen Lösungen der linearen Differentialgleichungen zweiter Ordnung*, Arkiv för Matematik, Astronomi och Fysik, vol. 12 (1917), No. 14.

For the exceptional integral we may in fact state a little more, namely that it can not vanish in the right half-plane including the imaginary axis.

For these exceptional integrals we may give a general statement. Let

$$J(z) = - F(z)$$

where $F(z)$ is real and positive for real values of $z > x_1$ and analytic in a region D including the real axis for $\Re(z) > x_1$. Concerning D we make the further assumption that any horizontal line which cuts D shall cut its boundary in one finite point only, and along the part of the line which lies in D one of the two relations

$$\Re[F(z)] > 0 \quad \text{or} \quad \Im m[F(z)] \neq 0$$

shall hold. Let $W(z)$ be a solution of

$$w'' - F(z)\, w = 0$$

such that $W(z) \longrightarrow 0$ when $z \longrightarrow \infty$ in D along a parallel to the real axis. Such an exceptional integral will under very general assumptions on $F(z)$ tend to zero so rapidly that the integrals in formulas (3.33) and (3.34) converge when the upper limit tends to ∞. This will for instance be the case when $F(z)$ is a polynomial. We conclude that *such an exceptional integral can not admit a finite zero in D nor can its derivative vanish there.*

The results of this section can be considerably extended in various directions. To every theorem we have stated for equations of type (2.242) corresponds a theorem for equations of general self-adjoint form. The reader can easily work out these theorems for himself.

3.4. **The star.** In the preceding section we have considered parallel lines. Let us now take a pencil of lines through a finite point $z = a$ where $J(z)$ is regular and does not vanish. Put

$$z = a + r\, e^{i\theta},$$
$$L(z) = (z - a)^2 J(z) = P(z) + iQ(z),$$

and draw the curves $P(z) = 0$ and $Q(z) = 0$. These curves intersect at $z = a$ where each of them has a double point, the tangents of which are given by

$$\cos 2\theta\, g_1(a) - \sin 2\theta\, g_2(a) = 0,$$
$$\cos 2\theta\, g_2(a) + \sin 2\theta\, g_1(a) = 0,$$

respectively. Let us follow a line in general position from $z = a$. We start with definite signs of $P(z)$ and $Q(z)$, and travel along the line until $Q(z)$ changes sign. If perchance $P(z)$ has been positive or has changed its sign, we break off

the line at that point. In case, however, $P(z)$ has been negative all along the segment, we continue to follow the line until finally even $P(z)$ changes sign, at which point we stop. We always stop at a point where $J(z)$ is singular no matter what the signs may be. In the same manner we treat every line of the pencil and determine on each ray a last point p_θ. The assemblage of all segments (a, p_θ) form together a configuration which we shall call *the star belonging to a.** The boundary of the star is made up of the points p_θ and parts of certain critical rays which are to be found among the tangents that can be drawn from the point $z = a$ to the curves $P(z) = 0$ and $Q(z) = 0$.

The points p_θ are either points of these curves or singular points of $J(z)$.

The appearance of the zero-free star is somewhat different from that of a star of convergence in as much as it in general is composed of two different regions which only touch each other at $z = a$. (See the figure below.) In the

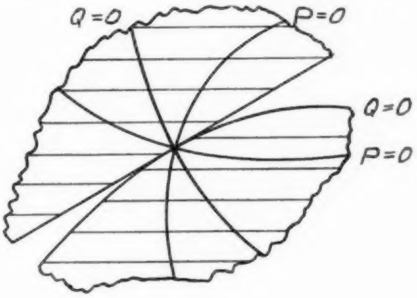

FIG. 1.

neighborhood of this point the boundary of the star is made up of that branch of the curve $Q(z) = 0$ which approaches $z = a$ through a region where $P(z) > 0$, and of the tangent to that branch at $z = a$. The fundamental property of the star is given by the following

THEOREM 3.4. *If $z = a$ is a zero of $w\,dw/dz$ where $w(z)$ is a solution of equation (2.242), then this product does not vanish throughout the star belonging to $z = a$, including the regular points of the boundary.*

This theorem is an immediate consequence of Theorem 3.31.†

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* It is, of course, the similarity in form and generation to the *star of convergence*, introduced in analysis by Prof. Mittag-Leffler, which suggests this name. The chief property of our star is indicated by the more explicit name *zero-free star of $w(z)$ with respect to $z = a$*.

† For further developments of the properties of the zero-free star cf. the paper quoted in section 3.9 below, also *Convex distribution of the zeros of Sturm-Liouville functions*, Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 261–65 and *A correction*, ibid., vol. 28 (1922), p. 462.

Let us use this theorem for proving that the entire function $E(z)$, considered in the preceding section, as well as its first derivative, has only real and negative zeros. $E(0) = 0$, but the origin is a singular point; we easily find however that the integrals in formulas (3.33) and (3.34) converge when z_1 is allowed to approach 0. Hence we can form the star for the point $z = 0$ and apply Theorem 3.4. We find $L(z) = -z$ and the star consists of the whole plane, except for the negative part of the real axis, which proves our assertion. What we have been proving is, of course, only that $J_1(u)$ has only real zeros, which is a well known fact in the theory of Bessel's functions.

3.5. **The primary domain.** Let us separate reals and imaginaries in formula (2.241). We obtain

$$\mathfrak{R}\left[\bar{w}\,\frac{dw}{dz}\right]_{z_1}^{z_2} - \int_{z_1}^{z_2}\left|\frac{dw}{dz}\right|^2 dx + \int_{z_1}^{z_2}|w|^2(g_1dx - g_2dy) = 0, \tag{3.51}$$

$$\mathfrak{Im}\left[\bar{w}\,\frac{dw}{dz}\right]_{z_1}^{z_2} + \int_{z_1}^{z_2}\left|\frac{dw}{dz}\right|^2 dy + \int_{z_1}^{z_2}|w|^2(g_1dy + g_2dx) = 0. \tag{3.52}$$

Assume $z = z_0$ is a zero of $w\; dw/dz$ lying in a domain T where $g_1(z)$ amd $g_2(z)$ are different from zero and $J(z)$ is regular. In order to have a definite case to deal with we suppose

$$g_1(z) > 0; \qquad g_2(z) < 0$$

in T. Draw parallels to the axes through z_0 and fix the attention on the second of the resulting quadrants of T. The points of this region are of two different kinds; a point α may be joined with z_0 by a curve which never increases with increasing values of $x = \mathfrak{R}(z)$, or a point may be such that there is no such curve. The points of the first kind form a certain simply connected region D_2 where the subscript 2 denotes the quadrant. Similarly we find a certain region D_4 in the fourth quadrant, the points of which all can be reached from z_0 by a continuous, never-increasing curve. These two regions together form a region $D_{24}(z_0)$ which we agree to call the primary domain of z_0. In view of (3.51) and (3.52) we obtain

THEOREM 3.5. *If there is a solution $w(z)$ of equation (2.242) such that $w(z)dw/dz$ vanishes at a point z_0 in a region T where $J(z)$ is analytic and $\mathfrak{R}[J(z)] \neq 0$, $\mathfrak{Im}[J(z)] \neq 0$, then neither $w(z)$ nor dw/dz can vanish in the primary domain of z_0.*

The primary domain is a region D_{13} if $\mathfrak{Im}[J(z)] > 0$, but a region D_{24} when $\mathfrak{Im}[J(z)] < 0$.

The primary domain gives of course only a first rough approximation of the

zero-free region in the neighborhood of a given zero, but even that is useful for orientation.

We have assumed that z_0 is not a point of a curve $g_1(z) = 0$ or $g_2(z) = 0$. If z_0 however lies on such a curve, we can still find a similar zero-free domain. Suppose, for example, that a segment (a,b) of the real axis forms a part of $g_2(z) = 0$ and that $a < z_0 < b$; then we can form a primary domain which is of type D_{14} if $g_2(z) > 0$ above the segment and of type D_{23} in the contrary case.

The boundary of a $D_{\mu\nu}(z_0)$ is made up of parts of the curves $g_1(z) = 0$ and $g_2(z) = 0$ and of lines parallel to the axes. It is as a rule possible to cross these boundaries and continue the zero-free region beyond the limits of the primary domain. The formulas (3.5) indicate clearly the way of doing this; we shall take up this question in a more general form in the next section.

3.6. **The standard domain.** Let us take our differential system in the general form

$$\frac{dw_1}{dz} = \frac{1}{K(z)} w_2, \qquad \frac{dw_2}{dz} = -G(z)w_1, \tag{3.61}$$

with the corresponding Green's transform, split up into real and imaginary parts

$$\mathfrak{R}\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_1 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_1 = 0, \tag{3.621}$$

$$\mathfrak{Im}\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_2 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_2 = 0, \tag{3.622}$$

where as usual

$$d\mathbf{\Gamma} = d\mathbf{\Gamma}_1 + i\,d\mathbf{\Gamma}_2 = G(z)dz, \tag{3.631}$$

$$d\mathbf{K} = d\mathbf{K}_1 + i\,d\mathbf{K}_2 = \frac{dz}{K(z)}. \tag{3.632}$$

We assume $G(z)$ and $K(z)$ to be single-valued analytic functions of z, existing in the whole plane. Then the two nets $\mathbf{\Gamma}$ and $\mathbf{K}$ are uniquely determined; one and the same curve may however correspond to different values of the constant of integration, depending upon which branch of the, in general, many-valued functions $\mathbf{\Gamma}(z)$ or $\mathbf{K}(z)$ we are considering.

A solution of (3.61) is as a rule a many-valued function which undergoes a linear transformation when z describes a closed contour in the plane surrounding singular points of the system. These are the zeros $a_1, a_2, \ldots, a_n, \ldots$ of $K(z)$, the singular points $b_1, b_2, \ldots, b_n, \ldots$ of $G(z)$ and $K(z)$ and, in general, the point

at infinity. In order to have to deal with single-valued functions only, we can either join the singular points of the system by properly chosen cuts which z is not allowed to cross, or we can construct a Riemann surface with branch points at the points (a_n), (b_n) and ∞ on which every solution of (3.61) is single-valued. We use the latter scheme and imagine the $\mathbf{\Gamma}$-net and the $\mathbf{K}$-net represented on the Riemann surface.

Now take a point a on a particular leaf of the surface where we know that $W(z) = w_1(z)w_2(z) = 0$. In order to show that another point b on the surface cannot be a zero of $W(z)$ we try to join a and b by a suitable path of integration. Formulas (3.62) suggest the use of the two nets for this purpose. We define a path curve as a *standard path* if it fulfils the following conditions:

(i) *It does not pass through any singular point*;

(ii) *It is composed of a finite number of arcs of curves belonging to the two base nets* $\mathbf{\Gamma}$ *and* $\mathbf{K}$;

(iii) *All along the path one and the same of the following four characteristic inequalities is satisfied, namely*

$$(1)\ \begin{matrix} d\mathbf{\Gamma}_1 \leqq 0, \\ d\mathbf{K}_1 \geqq 0; \end{matrix} \qquad (2)\ \begin{matrix} d\mathbf{\Gamma}_1 \geqq 0, \\ d\mathbf{K}_1 \leqq 0; \end{matrix} \qquad (3)\ \begin{matrix} d\mathbf{\Gamma}_2 \geqq 0, \\ d\mathbf{K}_2 \geqq 0; \end{matrix} \qquad (4)\ \begin{matrix} d\mathbf{\Gamma}_2 \leqq 0, \\ d\mathbf{K}_2 \leqq 0. \end{matrix}$$

With the definition we have chosen the tangent of the path is continuous except at a finite number of points where arcs of different base-curves meet. We make the additional agreement that in a small neighborhood of such a point the two meeting arcs are replaced by an arc which joins continuously and has a continuous tangent. This can always be done in such a manner that the characteristic inequality of the path is preserved.

According as the path in question is characterized by the first, second, third or fourth set of inequalities, we call the path a standard path of the first, second, third or fourth kind, respectively

$$S\mathbf{K}_1^+,\ S\mathbf{K}_1^-,\ S\mathbf{K}_2^+ \text{ and } S\mathbf{K}_2^-.$$

Take a point a on the surface where $G(z)$ and $K(z)$ are analytic and $K(a) \neq 0$ and construct all standard paths which start at a. In this fashion we obtain a certain continuum of curves spread out on the surface. The set of points which belong to at least one of these paths forms a region on the Riemann surface which we call the *standard* domain $D(z_0)$ of z_0. We include the boundary points of $D(z_0)$ in the standard domain except the point z_0 itself and eventual singular points of the differential equation. The fundamental property of $D(z_0)$ is given by

THEOREM 3.6. *If $z = z_0$ is a zero of $W(z) = w_1(z)w_2(z)$ then $W(z)$ cannot vanish in the standard domain of z_0.*

We note that the primary domain serves as a kernel of the standard domain in case $K(z) = 1$.

In our typical example we have

$$\mathbf{K}(z) = z; \qquad \mathbf{\Gamma}(z) = -\log z.$$

The **K**-net consists of the lines parallel to the axes; the $\mathbf{\Gamma}_1$-family is made up of the circles with center at the origin and the $\mathbf{\Gamma}_2$-family of the straight lines through the same point. The solutions of (3.35) are single-valued on the Riemann surface of $\log z$.

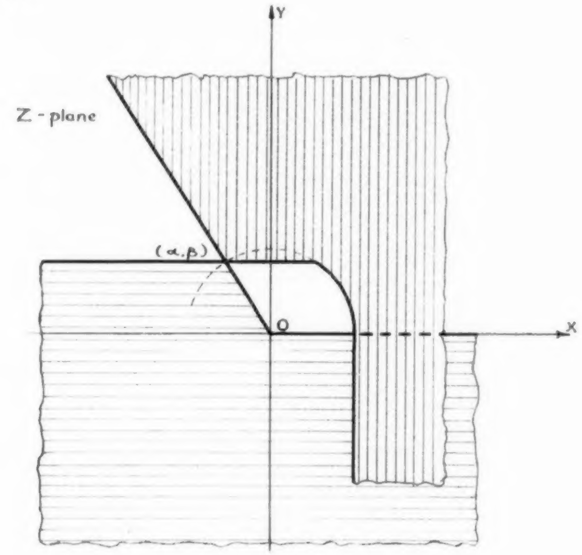

FIG. 2

Take a solution such that $W(x_0) = 0$ where $x_0 < 0$. A simple discussion shows that this function must be real for real negative values. If we construct the standard domain $D(x_0)$ we obtain the whole plane (except the origin) covered once and, in addition, two overlapping regions $\Re(z) \geqq |x_0|$, $\Im m(z) > 0$ and $\Re(z) \geqq |x_0|$, $\Im m(z) < 0$, attached to the lower (upper) side of the positive real axis. Using the method which is explained in the subsequent section, we could even prove a little more, namely: every solution which is real on the negative real axis has only real negative zeros in the sector $-(3\pi/2) < \arg(-z) < (3\pi/2)$.

Now take a solution for which $W(x_0) = 0$ with $x_0 > 0$. Such a solution must be real for real positive values of z. The standard domain of x_0 consists of

(1) the half-plane $\Re(z) \geqq x_0$ in one leaf of the surface;

(2) the region $|z| \leqq x_0$, $\Re(z) \geqq 0$ and $|\Im m(z)| \leqq x_0$, $\Re(z) < 0$ where the negative part of the real axis is counted as a double line; this all in the same leaf as (1);

(3) the quadrant $\Re(z) < 0$, $\Im m(z) > 0$ in another leaf, hanging on region (2) along the lower side of the real axis;

(4) the symmetric quadrant in a third leaf, hanging on region (2) along the upper side of the double line.

If the zero is not real we get standard domains which are asymmetric. The figure gives the typical shape of $D(a+ib)$ when $a < 0$, $b > 0$.

3.7. **The standard domain of a curve.** In the preceding sections we have formed various zero-free domains of a point. We can easily extend this conception to an arc of a curve.

Let C be an arc of a continuous curve on the surface, having a continuous tangent except at a finite number of points. We assume further that $G(z)$ is analytic along C and $K(z)$ analytic and different from zero. Let a solution of (3.35) be given along C. By given we understand that there is some means at our disposal by which we can ascertain the signs of the real and of the imaginary parts of $w\,(dw/dz)\,K(z)$.

Take a point z_0 on the curve C. From z_0 emanates a set of standard paths, belonging to all four families. Of these families we can as a rule only use two as paths of integration. Put

$$\overline{w(z)}\,\frac{dw}{dz}K(z) = U(z) + i\,V(z). \tag{3.71}$$

If, for example, $U(z_0) > 0$ we can use $S\mathbf{K}_1^+$ but not $S\mathbf{K}_1^-$. Thus we draw from every point on C the standard paths of the kinds which are indicated by the table below.

(3.72)

	negative	zero	positive
$U(z)$	$S\mathbf{K}_1^-$	$S\mathbf{K}_1^+, S\mathbf{K}_1^-$	$S\mathbf{K}_1^+$
$V(z)$	$S\mathbf{K}_2^+$	$S\mathbf{K}_2^+, S\mathbf{K}_2^-$	$S\mathbf{K}_2^-$

These paths together form a continuum of curves whose points make up the *standard domain of C with respect to $w(z)$*, which we denote by DCw. The points on the boundary of DCw are included in the standard domain by definition except the points on C and the singular points of the differential equation. In view of formulas (3.62) we have

THEOREM 3.7. *If a solution, $w(z)$, of* (2.11) *be given, in the sense defined above, along a curve C on the Riemann surface of $w(z)$, which curve does not pass through*

any of the singular points of the differential equation, then there is no zero of wdw/dz in the standard domain of C with respect to $w(z)$.

We have assumed that C does not pass through any of the singular points of the differential equation. This condition can, of course, be abandoned in special cases with great advantage. If, for instance, $z = a$ is a regular singular point with exponents λ_1 and λ_2 where $\Re(\lambda_1) > 1/2$ say, then the expressions in formulas (3.62) remain finite when z_1 is allowed to approach a, if $w_1(z)$ is the solution corresponding to λ_1. Further $w_1(z)w_2(z) \longrightarrow 0$ when $z \longrightarrow a$. Consequently the standard domain of such a point exists and has the same properties as the standard domain of an ordinary regular point.

In case $z = a$ is an irregular singular point we are often able to find certain sectors with vertex at a and corresponding solutions which tend toward 0 when $z \longrightarrow a$ within the sector and approach zero so rapidly that formulas (3.62) are applicable with a as the lower limit of integration. In this case we cán not construct a full standard domain around a, as we are allowed to use only such standard paths as, in a small neighborhood of a, lie on the sectors mentioned above. Such sectors can in general be found when, for instance, the irregular singular point is of finite rank.

3.8. **Change of variables.** In section 2.2 we have shown that the Green's transform is invariant under a transformation of the type

$$z = f(Z) \quad \text{or} \quad Z = F(z)$$

where $F(z)$ is, of course, an analytic function of z.

Take a point z_0 in the z-plane and form the corresponding standard domain $D(z_0)$. Suppose $F(z)$ is analytic in D and $F'(z) \neq 0$ there. Then $Z = F(z)$ will map $D(z_0)$ conformally on a region $d(Z_0)$ of the Z-plane. We assert that $d(Z_0)$ is the standard domain of Z_0 in the Z-plane. In fact, every standard path, $S\mathbf{K}$, starting from z_0 in the z-plane, is transformed into a curve, $s\kappa$, that starts at $Z_0 = F(z_0)$ in the Z-plane. In view of the formulas in section 2.2,

$$d\mathbf{\Gamma}(z) \equiv d\gamma(Z); \quad d\mathbf{K}(z) \equiv d\kappa(Z). \tag{3.81}$$

Hence the base nets in the two planes correspond to each other. From this it follows that $s\kappa$ is a standard path in the Z-plane, starting at Z_0 and, furthermore, $s\kappa$ is of the same kind as $S\mathbf{K}$. Thus every point of $d(Z_0)$ belongs to at least one standard path $s\kappa$; consequently $d(Z_0)$ is a standard domain as asserted. We can express this fact also by saying that *the standard domain of a point is an absolute covariant under conformal transformations of the plane.*

All the different forms of the Green's transform, given in section 2.2, are consequently equivalent for our purpose; one choice of independent variable

may however lead to simpler expressions for $k(Z)$ and $g(Z)$ or have some other advantage over another choice of variable. In the case of our typical equation

$$w'' - \frac{1}{z} w = 0$$

the transformation in formula (2.25) offers some advantages. By putting

$$Z = -\log z \quad \text{or} \quad z = e^{-Z},$$

the equation becomes

$$\frac{d}{dZ}\left[e^{Z} \frac{dw}{dZ}\right] - w = 0.$$

Every solution is now single-valued, in fact, an entire function of Z. This fact more than compensates the difficulties arising from the slightly more complicated form of the κ-net.

In order to get any new facts concerning the zero-free regions we must change both variables simultaneously. We shall not develop any generalities along this line. A simple example may be sufficient to show what use one can have of a linear transformation.

In section 3.6 we determined the standard domain corresponding to a solution of our typical equation with a positive zero or extremum. Let us indicate the (hypothetical) complex zeros of the principal branch of the solution in question by $z_1, z_2, \ldots, z_n, \ldots$ and $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n, \ldots$ where as usual $\bar{z}_n$ stands for the conjugate of z_n. Put $z_n = x_n + iy_n$. Our result showed only that from a certain value of n, $x_n < x_0$ and $y_n > x_0$. We shall prove in section 4.7 that $y_n = O(\sqrt{x_n})$. Thus it is desirable to obtain an upper limit for y_n for all values of n. This can be done as follows. Put

$$z = e^{i\theta} Z.$$

Equation (3.35) becomes

$$\frac{d^2w}{dZ^2} - \frac{e^{i\theta}}{Z} w = 0.$$

Thus we have

$$\gamma(Z) = -e^{i\theta} \log Z.$$

Putting $Z = Re^{i\phi}$ we get

$$\gamma_1(Z) = \sin\theta\,\phi - \cos\theta \log R,$$
$$\gamma_2(Z) = -\sin\theta \log R - \cos\theta\,\phi,$$

which shows that the γ-net is made up of logarithmic spirals. The point x_0 is transformed into $x_0e^{-i\theta}$; the corresponding γ_1- and γ_2-curves are

$$R = x_0e^{\tan\theta(\phi+\theta)},$$
$$R = x_0e^{-\cot\theta(\phi+\theta)},$$

respectively. The first curve has a vertical tangent at

$$\phi_0 = \theta, \quad R_0 = x_0e^{2\theta\tan\theta}.$$

It is easy to see that this point, as well as the points on the same vector for which $R > R_0$, belongs to a zero-free region. In order to determine the locus of R_0, ϕ_0 we introduce polar coördinates in the old z-plane by putting $z = re^{i\varphi}$. Then $\varphi = 2\theta$, $r = R$ and we obtain

$$r = x_0e^{\varphi\tan\frac{\varphi}{2}}.$$

Thus there is no zero of $w(z)\, dw/dz$ beyond this curve. Hence

$$y_n < 2\,|z_n|\ \arg\,(z_n)\,/\,\log\,|z_n|,$$

or

$$\arg z_n \longrightarrow \pi.$$

3.9. **The domain of influence.*** The zero-free domains considered in the preceding articles have all been obtained by specialization of the path of integration in formula (2.15). We can arrive at a more general type of zero-free region that embraces the previous ones as special cases in the following way.

The formula

$$(3.91)\qquad \Big[\overline{w_1}w_2\Big]_{z=z_2} - \Big[\overline{w_1}w_2\Big]_{z=z_1} - \int_{z_1}^{z_2} |w_2|^2 d\overline{\mathbf{K}} + \int_{z_1}^{z_2} |w_1|^2 d\,\mathbf{\Gamma} = 0$$

contains four terms. The first term will evidently not vanish if the sum of the other three terms is different from zero. This sum will certainly not be zero if the argument of each of the terms in the sum is known to lie in one and the same angle of opening less than π. The argument of the second term in (3.91) depends essentially upon the initial conditions. We have

$$(3.92)\qquad \arg\,(\overline{w_1}w_2) = \arg\,(dw/dz)/w + \arg K(z)$$

* This article was added at the revision of the paper in November, 1921.

where $(dw/dz)/w$ at $z = z_1$ is a known number λ which may be 0 or ∞. Put

$$\begin{aligned} \arg \lambda + \arg K(z_1) + \pi &= \iota_1, \\ \pi - \arg d\,\mathbf{K} &= \kappa_z, \\ \arg d\,\mathbf{\Gamma} &= \gamma_z. \end{aligned} \tag{3.93}$$

Given an angle θ such that

$$\theta < \iota_1 < \theta + \pi, \tag{3.931}$$

we can always choose the path of integration, starting from $z = z_1$, so that

$$\begin{aligned} \theta &< \kappa_z < \theta + \pi, \\ \theta &< \gamma_z < \theta + \pi, \end{aligned} \tag{3.932}$$

at all points z on the path C, say. Then it is obvious that the arguments of the third and the fourth terms in (3.91) fulfill the same condition as κ_z and γ_z do. Consequently the sum of the three latter terms in (3.91) is different from zero for this particular choice of the path of integration and

$$[\bar{w}_1 w_2]_{z=z_2} \neq 0 \tag{3.94}$$

for any point $z_2(\neq z_1)$ on C. We have of course tacitly assumed that the curve C does not pass through any of the singular points of the differential equation, an assumption that will be made throughout the present section.

To a given angle θ corresponds a set of paths of this nature which we call *the lines of influence with regard to the point* z_1 *and the solution* $w(z|z_1, \lambda)$ where z_1 and λ in an obvious manner denote the initial conditions. Giving θ all possible values, consistent with (3.931), we obtain all lines of influence belonging to $z = z_1$ and the solution in question. These lines we imagine to be traced on the Riemann surface mentioned above in section 3.6. The set of regular points on the surface which can be reached by lines of influence from the point $z = z_1$ forms a zero-free region, *the domain of influence of* z_1 *and* $w(z|z_1, \lambda)$ which we denote by $DI(z_1|\lambda)$.

It is evident that we obtain the most extensive domain for a given point z_1 when $w_1 w_2 = 0$ at $z = z_1$, i.e., when λ is 0 or ∞, there being no restriction (3.931) in that case on θ; and the regions we obtain when $\lambda \neq 0$ or ∞ are interior parts of $DI(z_1|0)$. Further we can verify that the standard domain and the star of the point z_1 are both contained in $DI(z_1|0)$.

In a similar manner we can define the domain of influence with regard to a curve and a solution.*

* For further developments of these ideas see the author's paper, *On the zeros of Sturm-Liouville functions*, Arkiv för Matematik, Astronomi och Fysik, vol. 16, No. (1922), 17.

4. The asymptotic distribution of the zeros*

4.1. **Reduction to the normal form.** In order to investigate the distribution of the zeros of a solution of a given differential equation in the neighborhood of an irregular singular point we use the transformation in formula (2.26). We have

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0. \tag{4.11}$$

By putting

$$Z(z|z_0) = \int_{z_0}^{z} \sqrt{\frac{G(z)}{K(z)}}\,dz, \tag{4.12}$$

we obtain

$$\frac{d}{dZ}\left[S(Z)\frac{dw}{dZ}\right] + S(Z)w = 0, \tag{4.13}$$

where

$$S(Z) = \sqrt{G(z)\,K(z)}. \tag{4.131}$$

As a parallel form of (4.13) we also use

$$\frac{d^2w}{dZ^2} + F(Z)\frac{dw}{dZ} + w = 0, \tag{4.14}$$

where

$$F(Z) = \frac{d}{dZ}\,[\log S(Z)]. \tag{4.141}$$

Let us further introduce a new independent variable

$$W = \sqrt{S(Z)}w. \tag{4.15}$$

The differential equation then becomes

$$\frac{d^2W}{dZ^2} + [1 - \phi\,(Z)]\,W = 0, \tag{4.16}$$

where

$$\phi(Z) = \frac{1}{2}\frac{dF}{dZ} + \frac{1}{4}\,[F(Z)]^2. \tag{4.161}$$

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* This section was rewritten at the revision of the paper. For the results compare with the memoir of P. Boutroux, *Recherches sur les transcendents de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre*, Annales de l'École Normale Supérieure, (3) vol. 30 (1913), pp. 255–375, and (3), vol. 31 (1914), pp. 99–159, especially §§2–6. Cf. further R. Garnier: *Sur les singularités irrégulières des équations différentielles linéaires*, Journal de Liouville, (8) vol. 2 (1919), pp. 99–200, especially §28.

The function $Z(z|z_0)$ depends upon an arbitrary parameter z_0 and, the value of z_0 being fixed, Z is in general infinitely-many-valued. We assume that we can find a determination of Z such that the function $\phi(Z)$ is single-valued and analytic in a region Δ of the Z-plane, extending to infinity and having the following properties:

CONDITION **A**

(1) Δ is simply-connected and smooth;

(2) Every line parallel to the real axis cuts the boundary Γ of Δ either (i) in a line-segment, or (ii) in a point, or (iii) not at all;

(3) Δ lies entirely in a sector

$$-\pi + \delta < \arg Z < \pi - \delta; \quad |Z| \geqq R_0 > 0.$$

A region Δ that fulfils a condition **A** is said to be of type **A.** In the case when Γ is actually cut by every parallel of the real axis we say the region is of type **Aa**, otherwise of type **Ab**. From the condition **A** it follows that Δ contains a strip Δ_0 given by some equality like

$$\Re(Z) \geqq A > R_0; \quad B_2 \geqq \Im m(Z) \geqq B_1. \tag{4.17}$$

Furthermore we assume that $\phi(Z)$ in Δ satisfies

CONDITION **B.**

$$|\phi(Z)| < \frac{M}{|Z|^{1+\nu}}$$

where ν and M are positive constants.

4.2. **An integral equation.** The method of successive approximations shows that the general solution of (4.15) is bounded in any strip Δ_0 of *finite width*. Consequently the expression

$$f(Z) = W_0(Z) + \int_Z^{\infty} \sin (T - Z)\, \phi\, (T)\, W(T)\, dT \tag{4.21}$$

has a definite meaning and represents an analytic function in Δ_0. Here $W_0(Z)$ is a solution of

$$W_0'' + W_0 = 0, \tag{4.22}$$

$W(Z)$ is a solution of (4.15) and the path of integration is a straight line parallel to the axis of reals. Moreover we find that

$$f''(Z) + f(Z) = \phi\, (Z) W(Z). \tag{4.23}$$

Consequently if $f(Z)$ is a solution of the integral equation.

$$(4.24)\qquad f(Z) = W_0(Z) + \int_Z^{\infty} \sin (T - Z)\phi(T) f(T)\, dT,$$

then $f(Z)$ is also a solution of (4.15). The equation (4.24) is a singular integral equation of Volterra's type which we can solve in the following way. Putting

$$(4.25)\qquad K_1(Z,T) = \sin (T - Z)\, \phi\, (T),$$

we define the iterated kernals

$$(4.251)\qquad \begin{aligned} K_2(Z, T) &= \int_T^{\infty} K_1(Z, U)K_1(U,T)\, dU, \\ &\cdots\cdots\cdots\cdots \\ K_n(Z, T) &= \int_T^{\infty} K_1(Z, U)K_{n-1}(U, T)\, dU, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and further

$$(4.252)\qquad E_n(Z) = \int_Z^{\infty} K_n(Z, T)\, W_0(T)\, dT.$$

Then

$$(4.253)\qquad W(Z) = W_0(Z) + \sum_{n=1}^{\infty} E_n(Z)$$

formally satisfies (4.24). If Δ_0 is a strip of finite width and T and Z are points in Δ_0, we have

$$(4.26)\qquad \begin{aligned} &|W_0(Z)| \leqq K, \\ &|\sin (T - Z)| \leqq K, \\ &|K_1(Z, T)| \leqq \frac{KM}{|T|^{\nu+1}}, \end{aligned}$$

and consequently

$$(4.261)\qquad |E_n(Z)| < \frac{K}{n!}\left(\frac{KM}{\nu}\right)^n \left|\frac{1}{Z}\right|^{n\nu}.$$

Hence the series in (4.253) is absolutely and uniformly convergent in Δ_0 and represents an analytic function of Z in that region, which function is furthermore a solution of the integral equation (4.24) and consequently also of the differential equation (4.15).

We know in advance that $W(Z)$ is bounded in Δ_0 and we find easily that

$$|W(Z)| < 2K, \tag{4.27}$$

when

$$|Z| > \left[\frac{2M}{\nu}\right]^{\frac{1}{\nu}} \text{ in } \Delta_0. \tag{4.271}$$

Using formula (4.24) we can improve this result and obtain a second approximation, namely

$$|W(Z) - W_0(Z)| < \frac{2K^2M}{\nu|Z|^\nu}, \tag{4.272}$$

when Z is subject to the condition in formula (4.271).

This shows that $W(Z)$ approaches $W_0(Z)$ indefinitely in Δ_0, or, using the terminology of Boutroux,* the function $W(Z)$ is asymptotic to $W_0(Z)$ in Δ_0.

Similarly we find

$$|W'(Z) - W_0'(Z)| < \frac{2K^2M}{\nu|Z|^\nu} \tag{4.273}$$

when Z satisfies (4.271); this shows that $W'(Z)$ is asymptotic to $W_0'(Z)$ in Δ_0.

4.3. **Oscillatory solutions and truncated ones.** By formula (4.253) we obtain to each solution $W_0(Z)$ of (4.22) a solution $W(Z)$ of the given differential equation (4.15) and $W(Z)$ is asymptotic to $W_0(Z)$. Taking a pair of linearly independent solutions $W_{01}(Z)$ and $W_{02}(Z)$ of (4.22) we obtain a pair of linearly independent solutions $W_1(Z)$ and $W_2(Z)$ of (4.15). In particular we choose

$$W_{01}(Z) = e^{iZ}; \quad W_{02}(Z) = e^{-iZ}. \tag{4.31}$$

The corresponding solutions $W_1(Z)$ and $W_2(Z)$ are asymptotic to e^{iZ} and e^{-iZ}, respectively, and consequently admit of no zeros in Δ_0 beyond a certain limit. The same is true of their derivatives. Even in the full region Δ these solutions do not admit of any zeros outside of a sufficiently large circle. We notice from formula (4.24) that

$$\begin{aligned} e^{-iZ}W_1(Z) &= 1 + \frac{\theta_1(Z)}{Z^\nu}, \\ e^{iZ}\,W_2(Z) &= 1 + \frac{\theta_2(Z)}{Z^\nu}, \end{aligned} \tag{4.311}$$

where $|\theta_k(Z)| < C$, a certain constant, when $|Z| > R$ in Δ, which proves the assertion. Using a term coined by Boutroux† we call these integrals *truncated in* Δ.

* Loc. cit., pp. 270–273.
† Loc. cit., pp. 261–263.

If the region Δ is of type **Aa** (vide supra) these two integrals are the only truncated ones in Δ. In fact, every solution of (4.15) can be represented in the form

$$W(Z)=c_1W_1(Z)+c_2W_2(Z),\quad c_1c_2\neq 0 \tag{4.32}$$

which is asymptotic to $W_0(Z)=c_1e^{iz}+c_2e^{-iz}$ in Δ. This region being of type **Aa** we can always choose the strip Δ_0, $B_1<\mathfrak{Im}(Z)<B_2$, in such a fashion that it contains the zeros of $W_0(Z)$, $\alpha_n=\alpha+n\pi$, from a certain $n=N_0$ on. As $c_1c_2\neq 0$ we can assume $W_0(Z)=\sin(Z-\alpha)$ without any loss of generality. Let us further denote the parts of Δ which lie above and below Δ_0 respectively by Δ_1 and Δ_{-1}. In these regions

$$W(Z)=\sin(Z-\alpha)+\frac{\theta_0(Z)}{Z^\nu},\ Z \text{ in } \Delta_0; \tag{4.330}$$

$$e^{+iZ}W(Z)=e^{+iZ}\sin(Z-\alpha)+\frac{\theta_1(Z)}{Z^\nu},\ Z \text{ in } \Delta_1; \tag{4.331}$$

$$e^{-iZ}W(Z)=e^{-iZ}\sin(Z-\alpha)+\frac{\theta_{-1}(Z)}{Z^\nu},\ Z \text{ in } \Delta_{-1}; \tag{4.332}$$

Here

$$|\theta_\lambda(Z)|<2K_\lambda^2M/\nu\quad(\lambda=-1,0,+1) \tag{4.333}$$

when

$$|Z|\geqq[2M/\nu]^{\frac{1}{\nu}} \tag{4.334}$$

in Δ_λ, where K_λ denotes the maximum of $|e^{\lambda iZ}\sin(Z-\alpha)|$ in Δ_λ.

Let us mark the points α_n in Δ_0 and surround each of them by a small circle Γ_n, $|Z-\alpha_n|=\epsilon$. On the circle Γ_n we have $|\sin(Z-\alpha)|>\frac{2}{\pi}\epsilon$. If we choose

$$|Z|\geqq R=[\pi K_0^2M/\nu\epsilon]^{\frac{1}{\nu}}, \tag{4.34}$$

we have $|\theta_0(Z)/Z^\nu|\leqq\frac{2}{\pi}\epsilon$. Hence the first term on the right hand side in (4.330) dominates the second one when Z is a point on a circle Γ_n in the part of Δ that lies outside of the circle Γ_R defined by (4.34). We can assume without considerable loss of generality that Γ_R cuts the boundary Γ of Δ in two points only. Then Γ_R divides Δ in two parts, one interior $\Delta-$ and one exterior $\Delta+$. We denote by Δ_* what is left of $\Delta+$ after punching out the interior of the circles Γ_n.

Then, *$W(Z)$ has one zero and only one in each of the circles Γ_n in $\Delta+$ and no zeros at all in Δ_*.* The first part of the statement is a consequence of a well

known theorem due to Rouché.* The latter part of the assertion depends upon the fact that $\sin(Z-\alpha)$ is the majorating term in the expression for $W(Z)$ in Δ_*, provided ϵ is small enough. In the same way we find that the zeros of $W'(Z)$ in $\Delta+$ all lie in the circles Γ'_n, $|Z-\alpha'_n| = \epsilon$ where $\alpha'_n = \alpha_n + \frac{\pi}{2}$, namely one in each circle.

Thus in the case of a region Δ of type **Aa** the zeros of $W(Z)$ are approximated by the zeros of a certain sine-function, $C\sin(Z-\alpha)$, to which $W(Z)$ is asymptotic, and the zeros of $W'(Z)$ are approximated by the zeros of $C\cos(Z-\alpha)$. Let the zeros of $W(Z)$ in Δ be denoted by A_n and those of $W'(Z)$ by A'_n where the notation is so chosen that

$$\lim (A_n - \alpha_n) = 0 \text{ and } \lim (A'_n - \alpha'_n) = 0.$$

Then we say that the points A_n form a *string of zeros in* Δ and similarly the points A'_n.† There are only two linearly independent solutions which do not have a string of zeros in Δ, namely the truncated solutions defined above.

In the case of a region Δ of type **Ab**, however, we have infinitely many independent solutions which are truncated in Δ, namely all for which the zeros of the asymptote-function $W_0(Z)$ lie outside of Δ. On the other hand, given a point set $\alpha_n = \alpha + n\pi$ which lies in Δ from a certain n on, we can always find a solution $W(Z)$ of (4.15) which is asymptotic to $\sin(Z-\alpha)$ in Δ and the zeros of which form a string in Δ, approximated by (α_n). This solution is given by our previous formulas (4.25) with $W_0(Z) = \sin(Z-\alpha)$.

In order to emphasize the dependence of the solution $W(Z)$ on the parameter α we denote it by $W(Z,\alpha)$. The zeros of $W(Z,\alpha)$ in Δ are functions of α which justifies the writing of $A_n(\alpha)$ instead of simply A_n. Let us see how $A_n(\alpha)$ varies with α. We put $\alpha = \sigma + i\tau$ and let σ increase from σ_0 to $\sigma_0 + \pi$, keeping τ constant $= \tau_0$. Then $A_n(\alpha)$ describes a certain curve from $A_n(\alpha_0)$ to $A_n(\alpha_0 + \pi) = A_{n+1}(\alpha_0)$, which shows the relationship between the functions $A_n(\alpha)$. The path followed by $A_n(\alpha)$ is almost a straight line from a point close to $\alpha_n = \alpha_0 + n\pi$ to a point close to α_{n+1}. If σ continues to increase, $A_n(\alpha)$ will describe a certain curve $\Sigma(\tau_0)$ that joins the different zeros in the string, belonging to $W(Z,\alpha_0)$. We call this curve a *zero-curve* (of solutions) of the differential equation (4.15). The curve is evidently asymptotic to the line $\mathfrak{Im}(Z) = \tau_0$ and is uniquely characterized by its asymptote. Through every point in $\Delta+$ there passes one and only one zero-curve.

4.4. **Varied conditions.** In the preceding articles we have assumed $\phi(Z)$ to

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* Rouché, *Mémoire sur la série de Lagrange*, Journal de l'École Polytechnique, cahier 39 (1862), pp. 217–218.

† Boutroux and Garnier both use the term *ligne de zéros* for what we denote by *string of zeros*.

be analytic in an infinite region in the right half-plane. The case when $\phi(Z)$ is analytic and fulfils a condition **B** in a region Λ in the left half-plane of type **A** (with suitable change of condition **A**(3)) can be treated in a similar manner. We find that the general solution of (4.15) is asymptotic to some sine-function $W_0(Z)$ in Λ and there are solutions the zeros of which form a string asymptotic to a given point set $\alpha_{-n} = \alpha - n\pi$ in Λ.

A special case of large importance is when $\phi(Z)$ is analytic and fulfils a condition **B** in an upper half-plane Υ, defined by $\mathfrak{Im}(Z) > B_2$. In this case every solution is asymptotic to some sine-function $W_{0\alpha}(Z)$ in Υ_+, the extreme right part of Υ, and asymptotic to some other sine-function $W_{0\beta}(Z)$ in Υ_-, the extreme left part of Υ. Thus the solution may be truncated in Υ_+ or in Υ_- or in both of these regions. If the solution in question is oscillatory in Υ_+ then the zeros form one and only one string which is approximated by a point set (α_n). And if the solution is oscillatory in Υ_- then the zeros form a single string that approaches a point set (β_{-n}) indefinitely.

On the other hand given a sine-function $W_{0\alpha}(Z)$ there exists a solution $W_+(Z)$ of (4.15) that is asymptotic to $W_{0\alpha}$ in Υ_+, and there is another solution $W_-(Z)$ that is asymptotic to $W_{0\alpha}$ in Υ_-. But if $\tau = \mathfrak{Im}(\alpha)$ is a very large number then there exists a solution $W_\alpha(Z)$ that is asymptotic to $\sin(Z - \alpha)$ throughout Υ. The zeros of $W_\alpha(Z)$ are approximated by $\alpha \pm n\pi$ and consequently the two strings in Υ_+ and in Υ_- join into one doubly unlimited string. There are evidently no zeros above this string and only a finite number below it in Υ.

We can of course treat a lower half-plane in a similar manner.

4.5. **The distribution in the z-plane.** Let us now try to pass back to the z-plane. We will be chiefly concerned with the case of a region Δ in the right half-plane. The function

$$Z = \int_{z_0}^{z} \sqrt{\frac{G(z)}{K(z)}}\, dz \tag{4.51}$$

establishes a one-to-one and conformal correspondence between the points of the interior of the region Δ in the Z-plane and the interior points of a certain region D of the z-plane. This latter region is of course simply connected but may overlap itself infinitely often. In the case of overlapping we consider D as part of a Riemann surface $\mathfrak{D}$ over the z-plane on which we proceed to study the integrals $w(z)$. These solutions are of course single-valued and analytic in the interior of D (on $\mathfrak{D}$) which does not contain any of the singular points of the differential equation (4.11). There is however at least one singular point on the boundary of D corresponding to $Z = \infty$.

By this conformal mapping, the zero-curves $\Sigma(\tau)$ of the differential equation (4.15) are transformed into a set of curves $S(t)$ in D. The points α_n are carried

over into points a_n in D; similarly the points A_n into points A_n. The circles Γ_n go over into closed contours C_n around a_n. Finally the circular arc Γ_R in Δ is carried over into a curve C_R in D which separates this region into two parts D_+ and D_- corresponding to Δ_+ and Δ_- respectively. The points of D_+ that remain after leaving out the interior of the closed contours C_n we denote by D_*.

To the solution $W(Z,\alpha)$ of (4.15) corresponds a solution $w(z,a)$ of (4.11) by the relation

$$w(z, a) = [S(Z)]^{-\frac{1}{2}}W(Z, \alpha). \tag{4.52}$$

If α_n lies in Δ from a certain $n = N_0$ on, then the corresponding points a_n will finally lie in D. The solution $w(z,a)$ will have precisely one zero within each of the curves C_n in D_+, namely at A_n, and there will be no zeros at all in D_*. If D corresponds to a region Δ of type **Aa** there are two and only two linearly independent solutions which are truncated in D, namely those corresponding to $W_1(Z)$ and $W_2(Z)$, and every other solution oscillates infinitely often in D_+ When the corresponding region Δ is of type **Ab,** however, there are infinitely many solutions which are truncated in D.

The zero-curves of the equation (4.11) as far as the region D is concerned are precisely the curves S mentioned above. These curves are asymptotic to the curves $\mathfrak{S}$ satisfying the differential equation

$$\mathfrak{Im}\left\{\sqrt{\frac{G(z)}{K(z)}}dz\right\} = 0. \tag{4.53}$$

The integral curves of this equation we shall call the *asymptotic zero-curves* of (4.11).

Through every point a in D (on $\mathfrak{D}$) passes one curve $\mathfrak{S}$ and only one. Follow this curve $\mathfrak{S}(a)$ in the direction of increasing values of $\mathfrak{R}(Z)$ and mark the points $a_1, a_2, \ldots, a_n, \ldots$ defined by the equalities

$$n\pi = \mathfrak{R}\left\{\int_a^{a_n}\sqrt{\frac{G(z)}{K(z)}}\,dz\right\} \qquad (n = 1, 2, \ldots), \tag{4.54}$$

the integral taken along $\mathfrak{S}(a)$.

By our previous investigation we know that there is a definite solution $w(z,a)$ of (4.11) having the points a_n as limiting points of its zeros in the sense that the actual zeros A_n can be so numbered that $\lim (A_n - a_n) = 0$. Let us draw two circles Γ_{R_1} and Γ_{R_2} in the Z-plane with radii $R_2 > R_1 > R$. Suppose for the sake of simplicity that these circles cut Γ, the boundary of Δ, each in two points only. The maps in the z-plane of these circular arcs are two curves C_{R_1} and C_{R_2} in D which together with the boundary of D enclose a curvilinear quadrilateral

$[D]$ on $\mathfrak{D}$. This region is cut in two points only by $\mathfrak{S}(a)$, z_1 on C_{R_1} and z_2 on C_{R_2} say. Then the number of zeros of $w(z, a)$ in $[D]$ is given by

$$N[D] = \frac{1}{\pi}\Re\left\{\int_{z_1}^{z_2}\sqrt{\frac{G(z)}{K(z)}}\,dz\right\} + \theta,^* \tag{4.55}$$

where the integral is taken along $\mathfrak{S}(a)$ and $-1 \leqq \theta \leqq +1$.

Let us now consider the zeros of $w'(z,a)$. We have

$$\frac{dw}{dz} = \left[\frac{dW}{dZ} - \frac{1}{2}F(Z)W(Z)\right]\left[S(Z)\right]^{-\frac{1}{2}}\frac{dZ}{dz}. \tag{4.56}$$

The second and the third factors do not concern us. We note that $W(Z)$ is bounded in Δ_0 and assume that

$$|F(Z)| < \frac{M}{|Z|^{\mu}} \qquad (\mu > 0).^\dagger \tag{4.561}$$

Further, the function dW/dZ vanishes once in each of the circles Γ'_n and not at all in Δ'_*. Using the theorem of Rouché, quoted above, we conclude that the function $dW/dZ - \frac{1}{2}F(Z)W(Z)$ vanishes once and only once in each of these circles and is different from zero in every finite point of Δ_*, provided the quantity ϵ which enters in formula (4.34) is small enough. Hence dW/dZ vanishes once and only once in each of the corresponding closed contours C'_n in $D+$ and not at all in D'_*.

Just a few words about regions Λ and Υ considered in section 4.4. A region Λ of type **A** in the Z-plane corresponds to a region L in the z-plane. The zeros form a simple string just as in the case of a region D. A region Υ corresponds to a region U in the z-plane; this region consists of three parts $U+$, $U-$ and U° corresponding to $\Upsilon+$, $\Upsilon-$ and the intermediate portion of Υ respectively. The zeros form simple strings in $U+$ and $U-$ (one string in each at most); in certain cases these strings may join to one doubly unlimited string.

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* Compare the result in the real case given by Wiman, loc. cit., p. 7. Professor Wiman has informed me that he has used the more or less classical transformation in formula (4.12) in his lectures at the University of Upsala for studying the properties of solutions of linear differential equations of the second order in the complex plane. The general character of the distribution of the zeros in the neighbourhood of an irregular singular point has been known to him for some years.

† The function $F(Z)$ satisfies a Riccati equation (4.161). Using this equation it can be proved that $F(Z)$ will satisfy (4.561) in Δ with $\mu = 1$ or ν where ν is the exponent entering in condition **B**, provided $\nu \geqq 1$. When $0 < \nu < 1$ the general solution of (4.161) may admit strings of poles in Δ. Leaving out properly chosen regions around these poles, condition (4.561) is still satisfied.

4.6. **Polynomial coefficients.*** Let us assume that $K(z)$ and $G(z)$ are polynomials

(4.61) $$K(z) = z^k + \cdots; \quad G(z) = g_0 z^g + \cdots.$$

Infinity is an irregular singular point if $g - k \geqq -1$, which is the only interesting case in this connection.

In this case $Z(z|z_0)$ is in general an abelian integral of the third kind. In the neighborhood of infinity we have, putting $g - k + 2 = m$,

(4.62) $$Z = \frac{2}{m}\sqrt{g_0}\, z^{\frac{m}{2}}\left\{1 + \sum_1^{\infty}{}' \frac{c_n}{z^n} + c_{m/2}\frac{\log z}{z^{m/2}}\right\},$$

where the logarithmic term occurs only when m is an even number. Conversely we find

(4.621) $$z = \left[\frac{m}{2\sqrt{g_0}} Z\right]^{\frac{2}{m}}\left\{1 + \sum_{1,1}^{\infty} C_{nr} Z^{-\frac{2}{m}(n+r)} (\log Z)^r\right\},$$

the series being convergent in any sector in the neighborhood of infinity. Further we have

(4.63) $$F(Z) = \frac{G(z)K'(z) + G'(z)K(z)}{[G(z)]^{\frac{3}{2}}\,[K(z)]^{\frac{1}{2}}}.$$

Expanding in ascending powers of z we obtain

(4.631) $$F(Z) = \frac{g + k}{2\sqrt{g_0}}\, z^{-\frac{m}{2}}\left\{1 + \sum_1^{\infty}\frac{a_n}{z^n}\right\}.$$

Hence, substituting the expansion for z from formula (4.631)

(4.632) $$F(Z) = \frac{g+k}{m}\frac{1}{Z}\left\{1 + \begin{matrix}\text{double series of the same}\\ \text{type as in formula (4.621)}\end{matrix}\right\}.$$

The expansion for $\phi(Z)$ is of similar form but starts with a factor

$$[(3k - g + 4)/4m^2](1/Z^2).$$

Thus $\phi(Z)$ satisfies a condition **B** with $\nu = 1$ in any region outside of a sufficiently large circle in which the argument of Z is bounded. Let us take as our region Δ the part of the Z-plane in which $|Z| > R$ and $\Re(Z) \geqq 0$. If

* Cf. J. Horn, *Über die irregulären Integrale der linearen Differentialgleichungen zweiter Ordnung*, Acta Mathematica, vol. 23 (1900), pp. 171–201, especially p. 198.

R is sufficiently large this region is conformally mapped by $z(Z|Z_0)$ on a smooth sectorial region $D^{(\mu)}$ in the z-plane in which

$$\frac{1}{m}\left[(2\mu-1)\pi-\theta_0\right]-\delta < \arg z < \frac{1}{m}\left[(2\mu+1)\pi-\theta_0\right]+\delta \tag{4.64}$$

where δ is a small number, $\theta_0 = \arg g_0$ and $\mu = 0, 1, \ldots, m-1$ depending upon which determination of $\sqrt[m]{Z}$ we use in formula (4.621). The determination of the logarithm does not affect the form of the region appreciably.

The asymptotic zero-curves in the z-plane are given by

$$r^{\frac{m}{2}} \sin \frac{1}{2}(m\theta+\theta_0) + \cdots = \text{const.} \qquad (z = re^{i\theta}) \tag{4.65}$$

showing the leading term. They form m pencils in the neighborhood of infinity which point they approach in the directions

$$\theta^{(\mu)} = \frac{1}{m}\left[2\mu\pi-\theta_0\right] \qquad (\mu=0, 1, \ldots, m-1). \tag{4.651}$$

In general $w(z|a)$ is not single-valued in the neighborhood of infinity. Let $\mathfrak{T}$ be the part of the Riemann surface of $\log z$ which lies outside of a large circle. Then the solution is single-valued on $\mathfrak{T}$. The zeros of $w(z|a)$ are seen to form m strings asymptotic to the directions $\theta^{(\mu)}$ in *each* leaf of $\mathfrak{T}$. In particular if the solution is uniform in the neighborhood of infinity we find m strings in all.

If $N(r)$ denotes the number of zeros in a string within the circle $|z| = r$ and we put

$$n(r) = \frac{2}{m\pi}\sqrt{|g_0|}\, r^{\frac{m}{2}} \tag{4.66}$$

then $N(r)/n(r) \longrightarrow 1$ when $r \longrightarrow \infty$.

Let us take up the question of truncated solutions. By the results of section 4.3 we know that there are exactly two solutions which are truncated in one of the directions $\theta^{(\mu)}$. Hence the total number of truncated solutions is at most equal to $2m$. This number, however, actually reduces to m as we shall see.

Let us take a modified region $\overline{\Delta}$ bounded by a large circular arc $|Z| = R$, $\frac{\pi}{2} - \delta \geqq \arg Z \geqq -\frac{\pi}{2} + \delta$ (δ a small fixed positive quantity) and the tangents at the end-points of this arc, extended to infinity in the left half-plane. This is a region of type **A** in which $\phi(Z)$ satisfies a condition **B**. The two solutions, $W_1(Z)$ and $W_2(Z)$, truncated in $\overline{\Delta}$ are asymptotic to e^{iZ} and e^{-iZ}, respectively, in this region. Moreover they keep their asymptotic form in a wider

region, $W_1(Z)$ being asymptotic to e^{iZ} when $-\pi < \arg Z < 2\pi$ and $W_2(Z)$ being asymptotic to e^{-iZ} when $-2\pi < \arg Z < \pi$, as can be proved by considering a region $\overline{\Lambda}$ symmetric to $\overline{\Delta}$. Hence $W_1(Z) \longrightarrow 0$ in the upper half of $\overline{\Delta}$ and $W_2(Z) \longrightarrow 0$ in the lower half of $\overline{\Delta}$ provided $|\mathfrak{Im}(Z)| \longrightarrow \infty$. These solutions are furthermore uniquely determined by this property, as the solution of the integral equation (4.24) is unique.

The region $\overline{\Delta}$ corresponds to m different regions $\overline{D}^{(\mu)}$ in the z-plane in which

$$\theta^{(\mu-1)} + \epsilon \leqq \arg z \leqq \theta^{(\mu+1)} - \epsilon \qquad (\mu = 0, 1, \ldots, m-1). \tag{4.67}$$

These regions $\overline{D}^{(\mu)}$ of course have common parts; in fact, $\overline{D}^{(\mu)}$ has a part $U^{(\mu)}$, $\theta^{(\mu)} + \epsilon \leqq \arg z \leqq \theta^{(\mu+1)} - \epsilon$, in common with $\overline{D}^{(\mu+1)}$ and so on. By formula (4.52) we conclude that there is one solution truncated in $\overline{D}^{(\mu)}$ which tends towards zero in $U^{(\mu)}$ and a second solution truncated in $\overline{D}^{(\mu+1)}$ which approaches zero in the same region $U^{(\mu)}$. There is, however, only one solution which tends towards 0 in $U^{(\mu)}$. Thus we find the number of truncated solutions reduces to m; these solutions are uniquely characterized by their asymptotic properties: the solution $w_\mu(z)$ which approaches zero when $z \longrightarrow \infty$ in $U^{(\mu)}$ is truncated in the adjacent directions $\theta^{(\mu)}$ and $\theta^{(\mu+1)}$. It is worth while observing that the truncated solutions preserve the same asymptotic representation in three adjacent regions $U^{(\kappa)}$, namely $U^{(\mu-1)}$, $U^{(\mu)}$ and $U^{(\mu+1)}$ in the case of $w_\mu(z)$. This number m of truncated solutions is actually reached as is shown by considering the differential equations $w'' + z^g w = 0$.

Let us take the case when $K(z) \equiv 1$. Then the general solution of the differential equation which is an entire function of z can be represented in the form

$$w_\lambda(z) = w_1(z) - \lambda w_2(z), \tag{4.68}$$

where $w_1(z)$ and $w_2(z)$ are linearly independent solutions and λ is a complex parameter. When is this solution truncated? The answer is: λ *has to be one of the asymptotic values of the meromorphic function* $\lambda(z) = w_1(z)/w_2(z)$, or, in other words, λ *has to be a transcendental singular point of the inverse function* $z(\lambda)$. A quantity a is said to be an asymptotic value of an entire or meromorphic function $f(z)$ if there is a path L (a Jordan curve) tending to infinity along which $f(z)$ tends towards a. The condition is necessary because, if $w_\lambda(z)$ is a truncated solution, then we can find paths L along which $w_\lambda(z)/w_2(z) \longrightarrow 0$. Hence $w_1(z)/w_2(z) \longrightarrow \lambda$ along L; thus λ is an asymptotic value of $\lambda(z)$. The condition is also sufficient. If λ is an asymptotic value, $w_\lambda(z)/w_2(z) \longrightarrow 0$ along the corresponding path L. But

$$w_\lambda(z)/w_2(z) = W_\lambda(Z)/W_2(Z) \sim W_{0\lambda}(Z)/W_{02}(Z), \tag{4.69}$$

with obvious notation. This shows that the path can be taken outside of the sectors $\theta^{(\mu)} - \delta < \arg z < \theta^{(\mu)} + \delta$ and, moreover, that the limit is zero only if $W_{0\lambda}(Z)$ is either e^{iZ} or e^{-iZ}. But then $W_\lambda(Z)$ is truncated in $\bar{\Delta}$ and consequently $w_\lambda(z)$ is truncated in a corresponding region $\bar{D}^{(\mu)}$.

This result shows further that every function of the meromorphic family $[c_1w_1(z) + c_2w_2(z)]/[c_3w_1(z) + c_4w_2(z)]$ has at most m asymptotic values.

4.7. **Concluding remarks.** The discussion in this section has given us information on the distribution of the zeros in the neighborhood of an irregular singular point, in as much as we have means by which to determine the number of strings of zeros and their directions, and furthermore to find the restrictions imposed on the absolute value of the difference of two consecutive zeros in a string. This is often enough to give a fairly good picture of the general character of the distribution. The results in the third section, on the other hand, give information about zero-free regions around a given point or curve, and thus about the restrictions on the argument of the difference between two zeros. Hence the two methods are complementary.

The reader may find it instructive to work out for himself the distribution problems for the differential equation

$$w'' - \frac{1}{z}w = 0, \tag{4.71}$$

by combining the two methods. We find $Z = 2i\sqrt{z}$ and the transformed normal form is

$$W'' + \left(1 - \frac{3}{4Z^2}\right)W = 0. \tag{4.72}$$

The asymptotic zero-curves of (4.71) are parabolas with focus at the origin and the negative real axis as axis. The distribution of the zeros in the vicinity of the asymptotic parabola is very regular.

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ON CERTAIN RELATIONS BETWEEN THE PROJECTIVE THEORY OF SURFACES AND THE PROJECTIVE THEORY OF CONGRUENCES*

BY

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1. Introduction

Wilczynski has shown† that the system of partial differential equations

$$
(1)\qquad
\begin{aligned}
y_v &= mz, \quad z_u = ny,\\
y_{uu} &= ay + bz + cy_u + dz_v,\\
z_{vv} &= a'y + b'z + c'y_u + d'z_v,
\end{aligned}
$$

where

$$y_u = \frac{\partial y}{\partial u}, \quad y_v = \frac{\partial y}{\partial v}, \quad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \text{ etc.}$$

and where $m, n, a, \ldots, d'$ are functions of u and v, will be completely integrable if the following integrability conditions are satisfied:

$$
(2)\qquad
\begin{aligned}
&c = f_u,\ d' = f_v,\ b = -d_v - df_v,\ a' = -c'_u - c'f_u,\ mn - c'd = f_{uv},\\
&m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db',\\
&n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v = c'a + nb',\\
&2m_u n + mn_u = a_v + f_u mn + a'd,\\
&m_v n + 2mn_v = b'_u + f_v mn + bc'.
\end{aligned}
$$

In such a case, (1) will have precisely four linearly independent solutions $(y^{(K)}, z^{(K)})$ $(K = 1, 2, 3, 4)$. Let $y^{(1)}, \ldots, y^{(4)}$ and $z^{(1)}, \ldots, z^{(4)}$ be interpreted as the homogeneous coördinates of the two points P_y and P_z. As u and v vary, these points will describe two surfaces S_y and S_z and the line P_yP_z will generate a congruence whose focal surface consists of the two surfaces S_y and S_z.

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* Presented to the Society, under a different title, March 26, 1921.

† E. J. Wilczynski, *Sur la théorie générale des congruences*, Mémoires Publiés par la Classe des Sciences de l'Académie Royal de Belgique, Collection en 4,° ser. 2, vol. 3 (1911). This paper will hereafter be cited as the Brussels Paper.

The most general transformation which will leave the set (1) in the same form is

$$y = \lambda(u)\bar{y}, \quad z = \mu(v)\bar{z}, \quad \bar{u} = \alpha(u), \quad \bar{v} = \beta(v), \tag{3}$$

where λ, μ, α, β, are arbitrary functions of the variables indicated. The coefficients $\bar{m}$, $\bar{n}$, $\bar{a}$, ..., $\bar{d}'$ resulting from this transformation are

$$\begin{gathered}\bar{m} = \frac{\mu}{\lambda\beta_v} m, \quad \bar{n} = \frac{\lambda}{\mu\alpha_u} n, \quad \bar{d} = \frac{\mu\beta_v}{\lambda\alpha_u^2} d, \quad \bar{c}' = \frac{\lambda\alpha_u}{\mu\beta_v^2} c', \\ \bar{a} = \frac{1}{\alpha_u^2}\left(a + \frac{\lambda_u}{\lambda} c - \frac{\lambda_{uu}}{\lambda}\right), \bar{b} = \frac{\mu}{\lambda\alpha_u^2}\left(b + \frac{\mu_v}{\mu} d\right), \bar{c} = \frac{1}{\alpha_u}\left(c - 2\frac{\lambda_u}{\lambda} - \frac{\alpha_{uu}}{\alpha_u}\right), \\ \bar{a}' = \frac{\lambda}{\mu\beta_v^2}\left(a' + \frac{\lambda_u}{\lambda} c'\right), \bar{b}' = \frac{1}{\beta_v^2}\left(b' + \frac{\mu_v}{\mu} d' - \frac{\mu_{vv}}{\mu}\right), \bar{d}' = \frac{1}{\beta_v}\left(d' - 2\frac{\mu_v}{\mu} - \frac{\beta_{vv}}{\beta_v}\right);\end{gathered} \tag{4}$$

to which may also be added

$$\overline{(e)} = \frac{Ke^f}{\lambda^2\mu^2\alpha_u\beta_v} \tag{4'}$$

where K is an arbitrary constant.

There are various covariant configurations connected with the congruence. Some of these are connected with the focal surface and present themselves at first referred to a local coördinate system connected with that surface. Others appear directly in their relation to the local coördinate system of the congruence. In order that we may study the relations between these covariant configurations, their equations must be related to a common tetrahedron of reference. The fundamental covariants of the congruence form a tetrahedron, and the relations which exist between this tetrahedron and the local tetrahedra of reference for S_y and S_z, respectively, must therefore be obtained. The development of these relations constitutes the first part of this article.

There are two asymptotic curves which pass through a point P_y on S_y, and the osculating linear complexes of these two curves have a linear congruence in common. The two directrices of this congruence are called* (d_1') the *directrix of P_y of the first kind*; this directrix lies in the tangent plane of S_y at P_y but does not contain P_y, and (d_1'') the *directrix of P_y of the second kind*; this directrix passes through P_y but does not lie in the tangent plane.

The curves cut out on S_y by the parametric curves of the congruence form a conjugate system; the osculating planes at P_y of these two curves intersect in

* E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908), p. 95.

a line which passes through P_y and is called* (x_1) the *axis of the point* P_y with respect to the conjugate system.

The dual of the axis of P_y is called† (r_1) the *ray of* P_y with respect to the conjugate system.

There are four lines d'_2, d''_2, x_2, r_2 similarly connected with P_z on S_z making eight lines in all. In the second part of this paper, the equations of these eight lines referred to the tetrahedron of the congruence are obtained.

In the remaining parts of this article the possible coincidences of two or more of these lines are considered in general, and congruences which are characterized by certain pairs of coincidences are obtained in a canonical form. The existence of congruences which have no one of the eight associated lines indeterminate and which possess any one of twelve possible coincidences is established. In particular a study is made of the congruences which possess one of the following coincidence pairs: (1) the directrix of the first kind and the ray of P_y coincide, and the directrix of the second kind of P_y coincides with the ray of P_z, or (2) the axis of P_y coincides with the ray of P_z, and the ray of P_y coincides with the axis of P_z, or (3) the directrix of the first kind of P_y coincides with the directrix of the second kind of P_z, and the directrix of the second kind of P_y coincides with the directrix of the first kind of P_z. This class of congruence has a special interest since it can be connected with the theory of functions of a complex variable.

The writer wishes to thank Professor Wilczynski for his kindly interest and valuable suggestions.

2. Relations between the tetrahedra of reference of the focal sheet and the tetrahedron of the congruence

The differential equations which characterize S_y are‡

$$
\begin{aligned}
y_{uu} - \frac{d}{m}\, y_{vv} &= ay + cy_u + \left(\frac{b}{m} - \frac{d}{m}\frac{m_v}{m}\right) y_v, \\
y_{uv} &= m\,n\,y + \frac{m_u}{m}\, y_v.
\end{aligned} \tag{5}
$$

Make the transformation

$$u_1 = \varphi(u, v), \quad v_1 = \psi(u, v),$$

* E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 314.

† Loc. cit., pp. 317–18.

‡ E. J. Wilczynski, *Brussels Paper*, p. 44, equation 79.

where
$$\varphi_u = \delta\varphi_v, \quad \psi_u = -\ \delta\psi_v, \quad \delta^2 = -\frac{d}{m}.$$

The new parametric curves $u_1 =$ const. and $v_1 =$ const. will be the asymptotic curves of S_y and the transform of (5) will be in the intermediate form.*

Let $F(u, v)$ be a general function; then

$$(6) \qquad \begin{aligned} F_u &= \frac{\partial F}{\partial u} = \frac{\partial F}{\partial u}\frac{\partial \varphi}{\partial u} + \frac{\partial F}{\partial v_1}\frac{\partial \psi}{\partial u} = \delta(\varphi_v F_{u_1} - \psi_v F_{v_1}), \\ F_v &= \frac{\partial F}{\partial v} = \frac{\partial F}{\partial u_1}\frac{\partial \varphi}{\partial v} + \frac{\partial F}{\partial v_1}\frac{\partial \psi}{\partial v} = \varphi_v F_{u_1} + \psi_v F_{v_1}, \end{aligned}$$

whence

$$(7) \qquad \begin{aligned} F_{u_1} &= \frac{1}{2\delta\varphi_v}(F_u + \delta F_v), \\ F_{v_1} &= \frac{1}{2\delta\psi_v}(-F_u + \delta F_v). \end{aligned}$$

Using (6) and (7) one obtains

$$(8) \qquad \begin{aligned} y_u &= \delta(\varphi_v y_{u_1} - \psi_v y_{v_1}), \quad y_v = \varphi_v y_{u_1} + \psi_v y_{v_1}, \\ y_{uu} &= \delta^2(\varphi_v^2 y_{u_1u_1} - 2\varphi_v\psi_v y_{u_1v_1} + \psi_v^2 y_{v_1v_1}) + (\delta\varphi_{uv} + \delta_u\varphi_v)y_{u_1} \\ &\qquad\qquad - (\delta\psi_{uv} + \delta_u\psi_v)y_{v_1}, \\ y_{uv} &= \delta(\varphi_v^2 y_{u_1u_1} - \psi_v^2 y_{v_1v_1}) + (\delta\varphi_{vv} + \delta_v\varphi_v)y_{u_1} - (\delta\psi_{vv} + \delta_v\psi_v)y_{v_1}, \\ y_{vv} &= \varphi_v^2 y_{u_1u_1} + 2\varphi_v\psi_v y_{u_1v_1} + \psi_v^2 y_{v_1v_1} + \varphi_{vv}y_{u_1} + \psi_{vv}y_{v_1}. \end{aligned}$$

If these values (8) be substituted in (5), we find the differential equations of S_y referred to its asymptotic lines. These equations are

$$(9) \qquad \begin{aligned} y_{u_1u_1} + 2a_1 y_{u_1} + 2b_1 y_{v_1} + c_1 y = 0, \\ y_{v_1v_1} + 2a_1' y_{u_1} + 2b_1' y_{v_1} + c_1' y = 0, \end{aligned}$$

where

$$(10) \qquad \begin{aligned} 2a_1 &= \frac{1}{4\delta^2\varphi_v}\left(4\delta^2\frac{\varphi_{vv}}{\varphi_v} - \delta c - \frac{b}{m} - \delta^2\frac{m_v}{m} - 2\delta\frac{m_u}{m} + 3\delta\delta_v + \delta_u\right), \\ 2b_1 &= \frac{\psi_v}{4\delta^2\varphi_v^2}\left(- \delta c - \frac{b}{m} - \delta^2\frac{m_v}{m} - 2\delta\frac{m_u}{m} - \delta\delta_v - \delta_u\right), \\ c_1 &= -\frac{1}{4\delta^2\varphi_v^2}(a + 2\delta mn), \\ 2a_1' &= \frac{\varphi_v}{4\delta^2\psi_v^2}\left(- \delta c - \frac{b}{m} - \delta^2\frac{m_v}{m} + 2\delta\frac{m_u}{m} - \delta\delta_v + \delta_u\right), \\ 2b_1' &= \frac{1}{4\delta^2\psi_v}\left(4\delta^2\frac{\psi_{vv}}{\psi_v} + \delta c - \frac{b}{m} - \delta^2\frac{m_v}{m} + 2\delta\frac{m_u}{m} + 3\delta\delta_v - \delta_u\right), \\ c_1' &= -\frac{1}{4\delta^2\psi_v^2}(a - 2\delta mn). \end{aligned}$$

* E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Second Memoir), these Transactions, vol. 9 (1908).

Let us introduce the notations*

$$
(11)\quad
\begin{aligned}
B_1 &= \frac{1}{4}\left(c-2\frac{m_u}{m}-\frac{\delta_u}{\delta}\right)=\frac{1}{4}\frac{\partial}{\partial u}(f-\log \delta m^2)=\frac{1}{4}\left(f_u-\frac{1}{2}\frac{d_u}{d}-\frac{3}{2}\frac{m_u}{m}\right),\\
C_1 &= -\frac{1}{4}\left(\frac{b}{d}-\frac{m_v}{m}-\frac{\delta_v}{\delta}\right)=\frac{1}{4}\frac{\partial}{\partial v}(f+\log \delta^3 m^2)=\frac{1}{4}\left(f_v+\frac{3}{2}\frac{d_v}{d}+\frac{1}{2}\frac{m_v}{m}\right),\\
F_1 &= -\frac{1}{\delta}B_1-C_1,\\
G_1 &= \frac{1}{\delta}B_1-C_1;
\end{aligned}
$$

then equations (10) become

$$
(12)\quad
\begin{aligned}
2a_1 &= \frac{1}{\varphi_v}\left(\frac{\varphi_{vv}}{\varphi_v}+F_1-\frac{1}{\delta}\frac{m_u}{m}+\frac{\delta_v}{\delta}\right),\quad 2a_1' = \frac{\varphi}{\psi_v^2}F_1,\\
2b_1 &= \frac{\psi_v}{\varphi_v^2}G_1,\quad 2b_1' = \frac{1}{\psi_v}\left(\frac{\psi_{vv}}{\psi_v}+F_1+\frac{1}{\delta}\frac{m_u}{m}+\frac{\delta_v'}{\delta}\right),\\
c_1 &= -\frac{1}{4\delta^2\varphi_v^2}(a+2\delta mn),\quad c_1' = -\frac{1}{4\delta^2\psi_v^2}(a-2\delta mn).
\end{aligned}
$$

The fundamental semicovariants of S_y are†

$$
(13)\quad
\begin{aligned}
&y_1 = y,\quad z_1 = y_{u_1}+a_1y,\quad \rho_1 = y_{v_1}+b_1'y,\\
&\sigma_1 = y_{u_1v_1}+b_1'y_{u_1}+a_1y_{v_1}+\frac{1}{2}\left[(a_1)_{v_1}+(b_1')_{u_1}+2a_1b_1'\right]y,
\end{aligned}
$$

and the fundamental relative covariants of the congruence are‡

$$
(14)\quad y=y,\ z=z,\ \rho=y_u-\frac{m_u}{m}y,\ \sigma=z_v-\frac{n_v}{n}z.
$$

From (1), (7), (13) and (14) the following relations are obtained:

$$
(15)\quad
\begin{aligned}
y_1 &= y,\\
z_1 &= \left(a_1+\frac{1}{2\delta\varphi_v}\frac{m_u}{m}\right)y+\frac{m}{2\varphi_v}z+\frac{1}{2\delta\varphi_v}\rho,\\
\rho_1 &= \left(b_1'-\frac{1}{2\delta\psi_v}\frac{m_u}{m}\right)y+\frac{m}{2\psi_v}z-\frac{1}{2\delta\psi_v}\rho,\\
\sigma_1 &= P_1'y+Q_1'z+R_1'\rho+S_1'\sigma,
\end{aligned}
$$

* Cf. E. J. Wilczynski, *Brussels Paper*, p. 20, equation 17.

† E. J. Wilczynski, *Projective differential geometry of curved surfaces*, these Transactions, vol. 8 (1907), p. 248.

‡ E. J. Wilczynski, *Brussels Paper*, p. 23, equation 23.

where

$$
\begin{aligned}
4\delta^2\varphi_v\psi_v P_1 &= -a - \frac{m_u}{m}\left[\delta\left(\frac{\varphi_{vv}}{\varphi_v} - \frac{\psi_{vv}}{\psi_v}\right) + 2B_1\right] \\
&\qquad + 2\delta^2\varphi_v\psi_v[(a_1)_{v_1} + (b_1')_{u_1} + 2a_1b_1'], \\
4\varphi_v\psi_v Q_1 &= m\left(\frac{\varphi_{vv}}{\varphi_v} + \frac{\psi_{vv}}{\psi_v} - 6C_1 + 2\frac{d_v}{d} + 2\frac{n_v}{n}\right), \qquad (16)\\
4\delta^2\varphi_v\psi_v R_1 &= -\delta\left(\frac{\varphi_{vv}}{\varphi_v} - \frac{\psi_{vv}}{\psi_v}\right) - 2B_1\,, \\
4\varphi_v\psi_v S_1 &= 2m.
\end{aligned}
$$

Solving (15) for y, z, ρ and σ gives

$$
\begin{aligned}
y &= y_1\,, \\
z &= -\left(\frac{a_1\varphi_v + b_1'\psi_v}{m}\right)y_1 + \frac{\varphi_v}{m}z_1 + \frac{\psi_v}{m}\rho_1\,, \\
\rho &= -\delta\left(a_1\varphi_v - b_1'\psi_v + \frac{m_u}{\delta m}\right)y_1 + \delta\varphi_v z_1 - \delta\psi_v\rho_1\,, \qquad (17)\\
\sigma &= P_1 y + Q_1 z + R_1\rho + S_1\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 &= \frac{2\varphi_v\psi_v}{m}\left[-P_1' + \left(\frac{a_1\varphi_v + b_1'\psi_v}{m}\right)Q_1' + \delta\left(a_1\varphi_v - b_1'\psi_v + \frac{m_u}{\delta m}\right)R_1'\right], \\
Q_1 &= \frac{2\varphi_v\psi_v}{m}\left(-\frac{\varphi_v}{m}Q_1' - \delta\varphi_v R_1'\right), \\
R_1 &= \frac{2\varphi_v\psi_v}{m}\left(-\frac{\psi_v}{m}Q_1' + \delta\psi_v R_1'\right), \qquad (18)\\
S_1 &= \frac{2\varphi_v\psi_v}{m}\,.
\end{aligned}
$$

From equations (11) and (12) we find

$$
\begin{aligned}
a_1\varphi_v + b_1'\psi_v &= \frac{1}{2}\left(\frac{\varphi_{vv}}{\varphi_v} + \frac{\psi_{vv}}{\psi_v} - 2C_1 + 2\frac{\delta_v}{\delta}\right), \qquad (19)\\
a_1\varphi_v - b_1'\psi_v + \frac{m_u}{\delta m} &= \frac{1}{2}\left(\frac{\varphi_{vv}}{\varphi_v} - \frac{\psi_{vv}}{\psi_v} - \frac{2}{\delta}B_1\right),
\end{aligned}
$$

so that $P_1, \ldots, S_1$ may be written in the form

$$
(20)\quad
\begin{aligned}
P_1 = \frac{1}{m}\Big\{ & \frac{\varphi_{vv}}{\varphi_v}\frac{\psi_{vv}}{\psi_v} - \varphi_v\psi_v\Big[(a_1)_{v_1} + (b_1')_{u_1} + 2a_1b_1'\Big] + \frac{1}{2}\frac{\varphi_{vv}}{\varphi_v}\Big[\frac{m_u}{\delta m} - \frac{\partial}{\partial v}\Big(f + \log\frac{m}{n}\Big)\Big] \\
& - \frac{1}{2}\frac{\psi_{vv}}{\psi_v}\Big[\frac{m_u}{\delta m} + \frac{\partial}{\partial v}\Big(f + \log\frac{m}{n}\Big)\Big] - \frac{m}{d}\Big(\frac{a}{2} + \frac{m_u}{m}B_1 + B_1^2\Big) \\
& + \frac{1}{2}\Big(\frac{d_v}{d} + \frac{n_v}{n} - 3C_1\Big)\Big(\frac{d_v}{d} - \frac{m_v}{m} - 2C_1\Big)\Big\}, \\
Q_1 = & -\frac{\varphi_v}{m}\Big[\frac{\psi_{vv}}{\psi_v} - G_1 - \frac{1}{2}\frac{\partial}{\partial v}\Big(2f + \log\frac{md}{n^2}\Big)\Big], \\
R_1 = & -\frac{\psi_v}{m}\Big[\frac{\varphi_{vv}}{\varphi_v} - F_1 - \frac{1}{2}\frac{\partial}{\partial v}\Big(2f + \log\frac{md}{n^2}\Big)\Big], \\
S_1 = & \ \frac{2\varphi_v\psi_v}{m}.
\end{aligned}
$$

Let the coördinates of a point with respect to the tetrahedron $P_yP_zP_\rho P_\sigma$ of the congruence be x_1, x_2, x_3, x_4, and let the coördinates of the same point with respect to the tetrahedron $P_{y_1}P_{z_1}P_{\rho_1}P_{\sigma_1}$ of the surface S_y be x_1', x_2', x_3', x_4'; then

$$
(21)\qquad \omega(x_1'y_1 + x_2'z_1 + x_3'\rho_1 + x_4'\sigma_1) = x_1y + x_2z + x_3\rho + x_4\sigma ,
$$

where ω is a factor of proportionality which may be a function of u and v.

Substitution in (21) of the value of y, z, ρ and σ as given in (17) gives an identity in y_1, z_1, ρ_1, and σ_1 and leads to the following equations for the transformation of coördinates:

$$
(22)\quad
\begin{aligned}
\omega x_1' &= x_1 - \Big(\frac{a_1\varphi_v + b_1'\psi_v}{m}\Big)x_2 - \delta\Big(a_1\varphi_v - b_1'\psi_v + \frac{m_u}{\delta m}\Big)x_3 + P_1x_4 , \\
\omega x_2' &= \frac{\varphi_v}{m}x_2 + \delta\varphi_v x_3 + Q_1x_4 , \\
\omega x_3' &= \frac{\psi_v}{m}x_2 - \delta\psi_v x_3 + R_1x_4 , \\
\omega x_4' &= \frac{2\varphi_v\psi_v}{m}x_4.
\end{aligned}
$$

The computation for the second focal sheet is quite similar. The differential equations which characterize S_z are

$$
(23)\quad
\begin{aligned}
z_{vv} - \frac{c'}{n}z_{uu} &= b'z + \Big(\frac{a'}{n} - \frac{c'}{n}\frac{n_u}{n}\Big)z_u + d'z_v , \\
z_{uv} &= mnz + \frac{n_v}{n}z_u .
\end{aligned}
$$

Make the transformation

$$u_2 = \lambda(u,v) \ , \ v_2 = \mu(u,v) \ ,$$

where

$$\lambda_v = \epsilon\lambda_u, \quad \mu_v = -\epsilon\mu_u, \quad \epsilon^2 = -\frac{c'}{n} \ ,$$

in order to make the asymptotic curves of S_z, $u_2 = \text{const.}$ and $v_2 = \text{const.}$, parametric. Then one obtains

$$\begin{aligned}
z_u &= \lambda_u z_{u_2} + \mu_u z_{v_2} \ , \ z_v = \epsilon(\lambda_u z_{u_2} - \mu_u z_{v_2}) \ , \\
z_{uu} &= \lambda_{uu}^2 z_{u_2u_2} + 2\lambda_u\mu_u z_{u_2v_2} + \mu_u^2 z_{v_2v_2} + \lambda_{uu} z_{u_2} + \mu_{uu} z_{v_2} \ , \\
z_{uv} &= \epsilon(\lambda_u^2 z_{u_2u_2} - \mu_u^2 z_{v_2v_2}) + (\epsilon\lambda_{uu} + \epsilon_u\lambda_u) z_{u_2} - (\epsilon\mu_{uu} + \epsilon_u\mu_u) z_{v_2} \ , \\
z_{vv} &= \epsilon^2(\lambda_u^2 z_{u_2u_2} - 2\lambda_u\mu_u z_{u_2v_2} + \mu_u^2 z_{v_2v_2}) + (\epsilon\lambda_{uv} + \epsilon_v\lambda_u) z_{u_2} \\
&\qquad - (\epsilon\mu_{uv} + \epsilon_v\mu_u) z_{v_2}.
\end{aligned} \tag{24}$$

Equations (24) substituted in (23) give the differential equations of S_z referred to its asymptotic lines. These equations are

$$\begin{aligned}
z_{u_2u_2} &+ 2a_2 z_{u_2} + 2b_2 z_{v_2} + c_2 z = 0, \\
z_{v_2v_2} &+ 2a_2' z_{u_2} + 2b_2' z_{v_2} + c_2' z = 0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
2a_2 &= \frac{1}{\lambda_u}\left(\frac{\lambda_{uu}}{\lambda_u} + F_2 - \frac{n_v}{\epsilon n} + \frac{\epsilon_u}{\epsilon}\right) \ , \ 2a_2' = \frac{\lambda_u}{\mu_u^2} F_2, \ , \\
2b_2 &= \frac{\mu_u}{\lambda_u^2} G_2 \ , \ 2b_2' = \frac{1}{\mu_u}\left(\frac{\mu_{uu}}{\mu_u} + G_2 + \frac{n_v}{\epsilon n} + \frac{\epsilon_u}{\epsilon}\right) , \\
c_2 &= -\frac{1}{4\epsilon^2\lambda_u^2}(b' + 2\epsilon mn) \ , \ c_2' = -\frac{1}{4\epsilon^2\mu_u^2}(b' - 2\epsilon mn) \ ,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
B_2 &= -\frac{1}{4}\left(\frac{a'}{c'} - \frac{n_u}{n} - \frac{\epsilon_u}{\epsilon}\right) = \frac{1}{4}\frac{\partial}{\partial u}(f + \log\epsilon^3 n^2) = \frac{1}{4}\left(f_u + \frac{3}{2}\frac{c'_u}{c'} + \frac{1}{2}\frac{n_u}{n}\right), \\
C_2 &= \frac{1}{4}\left(d' - 2\frac{n_v}{n} - \frac{\epsilon_v}{\epsilon}\right) = \frac{1}{4}\frac{\partial}{\partial v}(f - \log\epsilon\, n^2) = \frac{1}{4}\left(f_v - \frac{1}{2}\frac{c'}{c'} - \frac{3}{2}\frac{n_v}{n}\right), \\
F_2 &= -\frac{1}{\epsilon} C_2 - B_2 \ , \\
G_2 &= \frac{1}{\epsilon} C_2 - B_2 \ .
\end{aligned} \tag{27}$$

The fundamental relative semicovariants of S_z are

$$\begin{aligned}
&y_2 = z, \quad z_2 = z_{u_2} + a_2 z, \quad \rho_2 = z_{v_2} + b_2' z, \\
&\sigma_2 = z_{u_2v_2} + b_2' z_{u_2} + a_2 z'_{v_2} + \frac{1}{2}\left[(a_2)_{v_2} + (b_2')_{u_2} + 2a_2 b_2'\right] z.
\end{aligned} \tag{28}$$

Using (14) and (28) one obtains

$$
(29)\qquad
\begin{aligned}
y &= -\left(\frac{a_2\lambda_u + b_2'\mu_u}{n}\right)y_2 + \frac{\lambda_u}{n} z_2 + \frac{\mu_u}{n}\rho_2 ,\\
z &= y_2 ,\\
\rho &= P_2y_2 + Q_2z_2 + R_2\rho_2 + S_2\sigma_2 ,\\
\sigma &= -\epsilon\left(a_2\lambda_u - b_2'\mu_u + \frac{n_v}{\epsilon n}\right)y_2 + \epsilon\lambda_u z_2 - \epsilon\mu_u\rho_2 ,
\end{aligned}
$$

where

$$
(30)\qquad
\begin{aligned}
P_2 = \frac{1}{n}\Big\{ & \frac{\lambda_{uu}}{\lambda_u}\frac{\mu_{uu}}{\mu_u} - \lambda_u\mu_u\left[(a_2)_{v_2} + (b_2')_{u_2} + 2a_2b_2'\right]\\
& + \frac{1}{2}\frac{\lambda_{uu}}{\lambda_u}\left[\frac{n_v}{\epsilon n} - \frac{\partial}{\partial u}\left(f + \log\frac{n}{m}\right)\right] - \frac{1}{2}\frac{\mu_{uu}}{\mu_u}\left[\frac{n_v}{\epsilon n} + \frac{\partial}{\partial v}\left(f + \log\frac{n}{m}\right)\right]\\
& - \frac{n}{c'}\left(\frac{b'}{2} + \frac{n_v}{n}C_2 + C_2^2\right) + \frac{1}{2}\left(\frac{c_u'}{c'} + \frac{m_u}{m} - 3B_2\right)\left(\frac{c_u'}{c'} - \frac{n_u}{n} - 2B_2\right)\Big\},\\
Q_2 = -\frac{\lambda_u}{\lambda}&\left[\frac{\mu_{uu}}{\mu} - G_2 - \frac{1}{2}\frac{\partial}{\partial u}\left(2f + \log\frac{c'n}{m^2}\right)\right],\\
R_2 = -\frac{\mu_u}{\mu}&\left[\frac{\lambda_{uu}}{\lambda_u} - F_2 - \frac{1}{2}\frac{\partial}{\partial u}\left(2f + \log\frac{c'n}{m^2}\right)\right],\\
S_2 = \ & \frac{2\lambda_u\mu_u}{n}.
\end{aligned}
$$

Let the coördinates of a point with respect to the tetrahedron $P_yP_zP_\rho P_\sigma$ of the congruence be x_1, x_2, x_3, x_4, and let the coördinates of this point with respect to the tetrahedron $P_{y_2}P_{z_2}P_{\rho_2}P_{\sigma_2}$ of the surface S_x be x_1'', x_2'', x_3'', x_4''; then

$$
(31)\qquad \pi(x_1''y_2 + x_2''z_2 + x_3''\rho_2 + x_4''\sigma_2) = x_1y + x_2z + x_3\rho + x_4\sigma,
$$

where π is a factor of proportionality which may be a function of u and v. If the values of y, z, ρ and σ as given in (29) are substituted in (31), there results an identity in y_2, z_2, ρ_2, and σ_2 which leads to the following relations:

$$
(32)\qquad
\begin{aligned}
\pi x_1'' &= -\left(\frac{a_2\lambda_u + b_2'\mu_u}{n}\right)x_1 + x_2 + P_2x_3 - \epsilon\left(a_2\lambda_u - b_2'\mu_u + \frac{n_v}{\epsilon n}\right)x_4,\\
\pi x_2'' &= \frac{\lambda_u}{n}x_2 + Q_2x_3 + \epsilon\lambda_u x_4,\\
\pi x_3'' &= \frac{\mu_u}{n}x_1 + R_2x_3 - \epsilon\mu_u x_4,\\
\pi x_4'' &= \frac{2\lambda_u\mu_u}{n}x_3.
\end{aligned}
$$

3. Equations of the eight lines referred to the tetrahedron of the congruence. A study of the possible coincidences

The equations of the directrix of the first kind of P_y referred to P_{y_1} P_{z_1} P_{ρ_1} P_{σ_1} are*

$$x_4' = 0, \quad x_1' + \frac{(a_1')_{u_1}}{2a_1'} + \frac{(b_1)_{v_1}}{2b_1}x_3' = 0. \tag{33}$$

If the values of $x_1', \ldots, x_4'$ as given by (22) are substituted in (33), the equations of the directrix of the first kind of P_y referred to $P_yP_zP_\rho P_\sigma$ are found to be

$$\begin{aligned} &x_4 = 0,\\ &x_1 + \frac{1}{m}\left[-(a_1\varphi_v + b_1'\psi_v) + \frac{(a_1')_{u_1}}{2a_1'}\varphi_v + \frac{(b_1)_{v_1}}{2b_1}\psi_v\right]x_2\\ &\qquad - \delta\left[a_1\varphi_v - b_1'\psi_v - \frac{m_u}{\delta m} - \frac{(a_1')_u}{2a_1'}\varphi_v + \frac{(b_1)_{v_1}}{2b_1}\psi_v\right]x_3 = 0. \end{aligned} \tag{34}$$

From equations (11) and (12) we find

$$\begin{aligned} \frac{(a_1')_{u_1}}{2a_1'}\varphi_v &= \frac{1}{4}\left[2\frac{\varphi_{vv}}{\varphi_v} + 3\frac{\delta_v}{\delta} + \frac{1}{\delta}\frac{\partial}{\partial u}\left(\log F_1\right) + \frac{\partial}{\partial v}\left(\log F_1\right)\right],\\ \frac{(b_1)_{v_1}}{2b_1}\psi_v &= \frac{1}{4}\left[2\frac{\psi_{vv}}{\psi_v} + 3\frac{\delta_v}{\delta} - \frac{1}{\delta}\frac{\partial}{\partial u}\left(\log G_1\right) + \frac{\partial}{\partial v}\left(\log G_1\right)\right], \end{aligned} \tag{35}$$

so that *the equations of the directrix of the first kind of $P_y(d_1')$ referred to* $P_yP_zP_\rho P_\sigma$ *become*

$$x_4 = 0, \quad x_1 + L_1x_2 + M_1x_3 = 0, \tag{36}$$

where

$$\begin{aligned} L_1 &= \frac{1}{4\delta m}\left[\frac{\partial}{\partial u}\log\frac{F_1}{G_1} + \delta\frac{\partial}{\partial v}\left(f + \log \delta^5 m^2 F_1G_1\right)\right],\\ M_1 &= \frac{1}{4}\left[\frac{\partial}{\partial u}\left(f + \log\frac{F_1G_1}{\delta m^2}\right) + \delta\frac{\partial}{\partial v}\log\frac{F_1}{G_1}\right]. \end{aligned} \tag{37}$$

The equations of the directrix of the second kind of P_y referred to $P_{y_1}P_{z_1}P_{\rho_1}P_{\sigma_1}$ are

$$x_2' + \frac{(b_1)_{v_1}}{2b_1}x_4' = 0, \quad x_3' + \frac{(a_1')_{u_1}}{2a_1'}x_4' = 0, \tag{38}$$

and these equations by the use of (11), (22) and (35) give as *the equations of the directrix of the second kind of P_y (d_1'') referred to $P_yP_zP_\rho P_\sigma$*

$$x_2 + S_1x_4 = 0, \quad x_3 + T_1x_4 = 0, \tag{39}$$

---

* E. J. Wilczynski, *Projective differential geometry of curved surfaces*, these Transactions, vol. 9 (1908), p. 95, equations (70a) and (70b).

where

$$
(40)\qquad
\begin{aligned}
S_1 &= \frac{1}{4\delta}\left[\frac{\partial}{\partial u}\log\frac{F_1}{G_1} + \delta\,\frac{\partial}{\partial v}\left(3f + \log\frac{\delta^7 m^2}{n^4}\,F_1G_1\right)\right],\\
T_1 &= \frac{1}{4\delta^2 m}\left[\frac{\partial}{\partial u}\,(f - \log \delta m^2 F_1G_1) - \delta\,\frac{\partial}{\partial v}\log\frac{F_1}{G_1}\right].
\end{aligned}
$$

In a similar way one may obtain *the equations of the directrix of the first kind of P_z (d_2') referred to $P_yP_zP_\rho P_\sigma$; they are*

$$
(41)\qquad x_3 = 0,\quad L_2x_1 + x_2 + M_2x_4 = 0,
$$

where

$$
(42)\qquad
\begin{aligned}
L_2 &= \frac{1}{4n}\left[\frac{\partial}{\partial u}\,(f + \log \epsilon^5 n^2 F_2G_2) + \frac{1}{\epsilon}\,\frac{\partial}{\partial v}\log\frac{F_2}{G_2}\right],\\
M_2 &= \frac{\epsilon}{4}\left[\frac{\partial}{\partial u}\log\frac{F_2}{G_2} + \frac{1}{\epsilon}\,\frac{\partial}{\partial v}\left(f + \log\frac{F_2G_2}{\epsilon n^2}\right)\right];
\end{aligned}
$$

and *the equations of the directrix of the second kind of $P_z(d_2'')$ referred to $P_yP_zP_\rho P_\sigma$, namely*

$$
(43)\qquad x_1 + S_2x_3 = 0,\quad T_2x_3 + x_4 = 0,
$$

where

$$
(44)\qquad
\begin{aligned}
S_2 &= \frac{1}{4}\left[\frac{\partial}{\partial u}\left(3f + \log\frac{\epsilon^7 n^2}{m^4}\,F_2G_2\right) + \frac{1}{\epsilon}\,\frac{\partial}{\partial v}\log\frac{F_2}{G_2}\right],\\
T_2 &= \frac{1}{4\epsilon n}\left[-\frac{\partial}{\partial u}\log\frac{F_2}{G_2} + \frac{1}{\epsilon}\,\frac{\partial}{\partial v}\left(f - \log \epsilon n^2 F_2G_2\right.\right].
\end{aligned}
$$

The equations of the axis of P_y (x_1) are*

$$
(45)\qquad x_3 = 0,\quad x_2 + K_1x_4 = 0,
$$

where

$$
(46)\qquad K_1 = f_v + \frac{d_v}{d} - \frac{n_v}{n}.
$$

The equations of the ray of $P_y(r_1)$ are†

$$
(47)\qquad x_1 = 0,\quad x_4 = 0.
$$

The equations of the axis of $P_2(x_2)$ are

$$
(48)\qquad x_4 = 0,\quad x_1 + K_2x_3 = 0,
$$

---

* E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 313.

† Loc. cit., p. 318.

where

$$K_2 = f_u + \frac{c'_u}{c'} - \frac{m_u}{m} \; ; \tag{49}$$

and of the ray (r_2)

$$x_2 = 0, \quad x_3 = 0. \tag{50}$$

Certain relations between the relative invariants B_1, C_1, B_2, C_2, K_1, $\ldots$, T_2 will be useful in the sequel, and are given here. They may be verified by computation:

$$\begin{aligned} S_1 - mL_1 &= K_1 - 2C_1 \, , \\ S_2 - nL_2 &= K_2 - 2B_2 \, , \\ 2B_1 - M_1 &= \delta^2 m T_1 \, , \\ 2C_2 - M_2 &= \epsilon^2 n T_2 \, . \end{aligned} \tag{51}$$

The points P_y and P_z will not coincide for all values of u and v, since we are excluding the case of congruences with coincident focal sheets.* Therefore the tangent plane to S_y at P_y $(x_4 = 0)$ is in general distinct from the tangent plane to S_z at P_z $(x_3 = 0)$. From these two facts one finds that certain pairs of lines cannot coincide (this may also be seen from the equations of the various lines); in fact, there remain only twelve possible coincidences. Let $(d'_1 d_2)$ denote the coincidence of the two lines d'_1 and d''_2, etc.; then the twelve possible coincidences and the corresponding analytic conditions are as follows:

$$\begin{array}{llll} (d'_1 d''_2): & L_1 = T_2 = S_2 - M_1 = 0; & (d'_2 d''_1): & L_2 = T_1 = S_1 - M_2 = 0; \\ (r_1 x_2): & K_2 = 0; & (r_2 x_1): & K_1 = 0; \\ (d'_1 r_1): & L_1 = M_1 = 0; & (d'_2 r_2): & L_2 = M_2 = 0; \\ (d'_1 x_2): & L_1 = M_1 - K_2 = 0; & (d'_2 x_1): & L_2 = M_2 - K_1 = 0; \\ (d''_1 x_1): & T_1 = S_1 - K_1 = 0; & (d''_2 x_2): & T_2 = S_2 - K_2 = 0; \\ (d''_1 r_2): & S_1 = T_1 = 0; & (d''_2 r_1): & S_2 = T_2 = 0; \end{array}$$

where the values of K_1, etc., are given in the preceding section.

The axis and ray of a point on either focal sheet of a congruence correspond by duality and the two directrices of such a point also correspond by duality. So *the coincidences* $(d''_1 x_1)$, $(d''_2 x_2)$, $(d''_1 r_2)$ *and* $(d''_2 r_1)$ *will be called the duals of* $(d'_1 r_1)$, $(d'_2 r_2)$, $(d'_1 x_2)$ *and* $(d'_2 x_1)$, *respectively. If a congruence has a coincidence then the congruence obtained from the given one by duality has the dual coincidence.* In fact, the substitution which transforms the coefficients of a congruence into the coefficients of the dual congruence transforms the conditions for any coincidence into the conditions for the dual coincidence. This fact enables one to reduce the amount of calculation involved in the study of the possible combina-

* E. J. Wilczynski, *Brussels Paper*, pp. 11–12.

tions of the twelve coincidences. For example, the canonical form and properties of the congruences characterized by the coincidences $(d_1'r_1)$ and $(d_2'r_2)$ gives by a substitution alone the canonical form and properties of the congruences characterized by the coincidences $(d_1''x_1)$ and $(d_2''x_2)$.

It is not evident that there exists a congruence which has any one of the twelve (apparently) possible coincidences, for there might be no solution to the integrability conditions and the corresponding coincidence conditions taken together. However, the congruence which has the coefficients

$$m = n = c' = d = 1, \quad a' = b = c = d' = 0, \quad a = -b' = \text{const.}, \tag{52}$$

satisfies the conditions for all the coincidences, so that there surely exists at least this one solution for any group of equations obtained by using any combination of the twelve coincidences. The congruences characterized by (52) will be referred to as *coincidence congruences* in the remainder of this article. Since $m \neq 0$, $n \neq 0$, neither of the focal sheets of a coincidence congruence degenerates into a curve, and since $c' \neq 0, d \neq 0$ neither of its focal sheets is developable.* We shall throughout consider only such $(mnc'd \neq 0)$ non-degenerate congruences. The coincidence congruences have four $(d_1', d_1'', d_2', d_2'')$ of their eight lines indeterminate, so that the coincidences in this case have but little meaning. We shall show that certain sets of coincidences give rise to congruences for which the eight lines are determinate, while for certain other sets some of the eight lines are of necessity indeterminate.

4. The properties and canonical form of congruences possessing coincidences $(d_1'r_1)$ and $(d_1''r_2)$

The conditions which give these coincidences are $L_1 = M_1 = S_1 = T_1 = 0$. From these equations and (51), we find

$$B_1 = 0, \quad K_1 = 2C_1, \quad F_1 = G_1 = -C_1. \tag{53}$$

Using (53) in $L_1 = 0$ and $M_1 = 0$ gives

$$\frac{\partial}{\partial v}\left(2f + \log \frac{d^5}{m} C_1^4\right) = 0, \quad \frac{\partial}{\partial u} \log C_1 = 0,$$

whence by integration

$$C_1 = V(v), \quad \frac{d^5}{m} C_1^4 e^{2f} = U(u)\ , \tag{54}$$

---

* E. J. Wilczynski, *Brussels Paper*, p. 28.

where $V(v)$ is an arbitrary function of v alone, and $U(u)$ is an arbitrary function of u alone.*

Integration of $C_1 = V(v)$ and $B_1 = 0$, using (11), gives

$$md^3e^{2f} = V_1(v)U_1(u), \quad md^3e^{-2f}V_2(v), \tag{55}$$

where $V_1(v)$ depends upon $V(v)$. From (54) and (55), one finds that

$$e^{4f} = \frac{U_1^2V_1^6}{UV_2},$$

so that $f_{uv} = 0$. Consider the most general transformation (3) which leaves the set (1) invariant; since $\overline{(e^f)} = \dfrac{Ke^f}{\lambda^2\mu^2\alpha_u\beta_v}$, we can choose λ, μ, α_u, β_v and K in such a way as to make

$$e^f = 1, \quad f = 0, \tag{56}$$

and this relation will be left unaltered by the subgroup of (3) for which $\lambda = \dfrac{c_1}{\sqrt{\alpha_u}}$, $\mu = \dfrac{c_2}{\sqrt{\beta_v}}$, where c_1 and c_2 are arbitrary constants.†

From (55) and (56) we find that $\dfrac{d}{m} = \sqrt{\dfrac{U_1V_1}{V_2}}$; since $\overline{\left(\dfrac{d}{m}\right)} = \dfrac{\beta_v^2}{\alpha_u^2}\dfrac{d}{m}$, we can make $d = m$, and thereafter use the subgroup of (3) for which $\alpha_u = \beta_v = c_1$, $\lambda = c_2$, $\mu = c_3$ where c_1, c_2 and c_3 are arbitrary constants.

We now have, using the fifth integrability condition,

$$f = 0, \quad d = m, \quad c' = n. \tag{57}$$

These values substituted in (54) give $mC_1 = \varphi(u)$; moreover we find by calculation that $C_1 = \dfrac{1}{2}\dfrac{m_v}{m}$, and from $B_1 = 0$ that $m = \psi(v)$ so that $\varphi(u) = \dfrac{1}{2}m_v$ while $m = \psi(v)$. Then $m = l_2v + l_1$ and $C_1 = \dfrac{l_2}{2(l_2v + l_1)}$.

* In the remainder of this article, arbitrary functions of u alone or of v alone are indicated by this same notation.

† Cf. E. J. Wilczynski, *Brussels Paper*, pp. 42–3. Similar transformations will often be made in the remainder of this article; in every case, the sub-group of (3) which is available for further transformations will be indicated.

There are two cases to consider.

First, $l_2 = 0$; then $m = \text{const.}$ and we find that the transformations at our disposal enable us to make

$$d = m = c' = n = 1. \tag{58}$$

Using (58) the integrability conditions give

$$a' = b = 0, \quad c = d' = 0, \quad b = -a = \text{const.}, \tag{59}$$

and (57), (58) and (59) characterize the coincidence congruences.

Second, $l_2 \neq 0$; now $\overline{(m)} = \dfrac{c_3}{c_1 c_2} m_1$ so we can make $l_2 = 1$, and thereafter use the subgroup of (3) for which

$$\alpha_u = \beta_v = c_1, \quad \lambda = c_2, \quad \mu = c_1 c_2.$$

Now $d = m = v + l_1$, $2C_1 = \dfrac{1}{v + l_1} = K_1 = \dfrac{\partial}{\partial v} \log \dfrac{d}{n}$ whence $c' = n = U_3(u)$. The sixth and seventh integrability conditions now become $a + b' = 0$ and $n_{uu} + n_{vv} = 0$ so that $n = c' = l_3 u + l_2$; but since $\overline{(c')} = \dfrac{1}{c_1^2} c'$ we can make $l_3 = 1$.* Using

$$f = c = d' = 0, \quad d = m = v + l_1, \quad c' = n = u + l_2 \tag{60}$$

in the integrability conditions, we obtain

$$a' = b = -1, \quad b' = -a, \tag{61}$$

where $a_v = 2(v + l_1)$, $a_u = -2(u + l_2)$, so that

$$a = v^2 - u^2 + 2(l_1 v - l_2 u) + l_3 , \tag{62}$$

where l_1, l_2 and l_3 are arbitrary constants.

If we use the values given by (60), (61) and (62), we obtain $L_2 = M_2 = S_2 = T_2 = 0$, $K_1 = \dfrac{1}{v + l_1}$, $K_2 = \dfrac{1}{u + l_2}$, so that these congruences possess the coin-

---

* If $l_3 = 0$, this cannot be done; but as in the first case above, we can show that if $l_3 = 0$ the congruence is a coincidence congruence.

cidences $(d_1'd_2'')$, $(d_2'd_1'')$, $(d_1''r_2)$ and $(d_2''r_1)$ in addition to the given pair. To recapitulate: *If a non-degenerate congruence possesses the coincidences* $(d_1'r_1)$ *and* $(d_1''r_2)$ *it is either a coincidence congruence which possesses all twelve coincidences, or else it is a congruence whose canonical form is*

$$(63)\quad \begin{aligned} f = c = d' = 0, \quad d = m = v + l_1, \quad c' = n = u + l_2, \\ a' = b = -1, \quad a = -b' = v^2 - u^2 + 2(l_1v - l_2u) + l_3 \end{aligned}$$

where l_1, l_2 *and* l_3 *are arbitrary constants, and possesses also the coincidences* $(d_1'd_2'')$, $(d_2'd_1'')$, $(d_1''r_2)$ *and* $(d_2''r_1)$.

The congruences dual to (63) have the canonical form*

$$(64)\quad \begin{aligned} f = c = d' = 0, \quad d = m = u + l_2, \quad c' = n = v + l_1, \\ b = a' = 0, \quad a = -b' = v^2 - u^2 + 2(l_1v - l_2u) + l_3, \end{aligned}$$

and possess the coincidences $(d_1'd_2'')$, $(d_2'd_1'')$, $(d_2'x_1)$, $(d_1'x_2)$, $(d_1''x_1)$ and $(d_2''x_1)$.

Since there exist congruences, other than the coincidence congruences, which possess either one of the sets of coincidences noted above, there exist congruences which possess any one of the coincidences separately. We will consider in the next section the congruences which possess the coincidences (r_1x_2) and (r_2x_1) and we shall show that there exist congruences, other than the coincidence congruences, which possess that pair of coincidences. Using that fact, we state that *there exist congruences, besides the coincidence congruences, which possess any one of the twelve possible coincidences.*

The coefficients given in (63) make $K_1 = \dfrac{1}{v + l_1} \neq 0$; if we seek those congruences which possess the coincidence (r_2x_1), which makes $K_1 = 0$, in addition to the coincidences $(d_1'r_1)$ and $(d_1''r_2)$, we find that they are coincidence congruences. More generally *the coincidence congruences are the only congruences which possess all of the twelve coincidences.* This justifies the term "coincidence congruences" which we have chosen for them.

5. The properties and canonical form of a congruence possessing coincidences (r_1x_2) and (x_1r_2)

Let us consider now a congruence which has the coincidences (r_1x_2) and (x_1r_2). By integration of the associated conditions

$$f_u + \frac{c'_u}{c'} - \frac{m_u}{m} = 0, \quad f_v + \frac{d_v}{d} - \frac{n_v}{n} = 0\,,$$

* E. J. Wilczynski, *Brussels Paper*, pp. 24–28.

one obtains

$$\frac{c'}{m} = V(v)e^{-f} , \quad \frac{d}{n} = U(u)e^{-f} .$$

Transformations (3) enable us by a proper choice of λ, μ, α_u, β_v to make $V(v) = 1$, $U(u) = 1$. We may use thereafter the sub-group of (3) given by

$$\lambda = \frac{1}{\sqrt{\alpha_u}} , \mu = \frac{1}{\sqrt{\beta_v}} .$$

We now have

$$c' = me^{-f}, \quad d = ne^{-f}. \tag{65}$$

Here, as previously, we exclude congruences for which $mnc'd = 0$.

From (65) we find

$$\begin{aligned} c'_u &= (m_u - mf_u)e^{-f}, & c'_{uu} &= (m_{uu} - 2m_uf_u - mf_{uu} + mf_u^2)e^{-f}, \\ d_v &= (n_v - nf_v)e^{-f}, & d_{vv} &= (n_{vv} - 2n_vf_v - nf_{vv} + nf_v^2)e^{-f}. \end{aligned}$$

If these values are substituted in the sixth and seventh integrability conditions, we obtain the equations

$$\begin{aligned} m_{uu} - m_uf_u - ma &= - e^{-f}(n_{vv} - n_vf_v - nb'), \\ m_{uu} - m_uf_u - ma &= - e^{f}(n_{vv} - n_vf_v - nb'); \end{aligned}$$

whence follows either $f = 0$, which means that the congruence is a W-congruence,* or else

$$\begin{aligned} m_{uu} - m_uf_u - ma &= 0 , \\ n_{vv} - n_vf_v - nb' &= 0 ; \end{aligned} \tag{66}$$

we shall call such congruences harmonic.†

If $f = 0$, equations (65) reduce to

$$c' = m, \quad d = n. \tag{67}$$

---

* E. J. Wilczynski, *Brussels Paper*, p. 46.

† The nets of curves u = const. and v = const. on S_y and S_z are in this case harmonic conjugate nets. See E. J. Wilczynski, *Geometrical significance of isothermal conjugacy of a net of curves*, American Journal of Mathematics, vol. 42 (1920), p. 215.

From (67) and the integrability conditions it follows that

$$
\begin{aligned}
&f = c = d' = 0, \quad b = -n_v, \quad a' = -m_u, \\
(68) \qquad &m_{uu} + n_{vv} = ma + nb', \\
&a_v = 3m_u n + mn_u, \quad b'_u = m_v + 3mn_v.
\end{aligned}
$$

The canonical form given here is the same as the canonical form for identically self dual congruences, or for congruences which belong to linear complexes.*

Using (66), we find

$$
\begin{aligned}
&c = f_u, \quad d' = f_v, \quad b = -n_v e^{-f}, \quad a' = -m_u e^{-f}, \quad mn(1 - e^{-2f}) = f_{uv}, \\
(69) \qquad &mn\left(3\frac{m_u}{m} + \frac{n_u}{n} - f_u\right) = \frac{m_{uuv}}{m} - \frac{m_{uu}}{m}\frac{m_v}{m} - \frac{m_{uv}}{m}f_u + \frac{m_u}{m}\frac{m_v}{m}f_u, \\
&mn\left(\frac{m_v}{m} + 3\frac{n_v}{n} - f_v\right) = \frac{n_{uvv}}{n} - \frac{n_{vv}}{n}\frac{n_u}{n} - \frac{n_{uv}}{n}f_v + \frac{n_v}{n}\frac{n_u}{n}f_v.
\end{aligned}
$$

The results of this section may be stated as follows:

If a non-degenerate congruence possesses the coincidences (r_1x_2) *and* (x_1r_2), *it is either a W-congruence whose canonical form is given by* (67) *and* (68) *in which case it is identically self-dual and belongs to a linear complex; or else it is an harmonic congruence whose canonical form is given by* (65), (66) *and* (69).

6. The properties and canonical form of congruences which possess coincidences $(d'_1d''_2)$ and $(d''_1d'_2)$

We consider finally the congruences which have the two coincidences $(d'_1d''_2)$ and $(d''_1d'_2)$. The conditions to be satisfied are

$$
(70) \qquad L_1 = T_2 = S_2 - M_1 = 0, \quad L_2 = T_1 = S_1 - M_2 = 0.
$$

Using (51), equations (70) may be replaced by

$$
(71) \quad L_1 = 0, \quad \frac{\partial}{\partial u}\log\frac{F_1}{G_1} + \delta\frac{\partial}{\partial v}\left(f + \log \delta^5 m^2 F_1 G_1\right) = 0\ ;
$$

$$
(72) \quad L_2 = 0, \quad \epsilon\frac{\partial}{\partial u}\left(f + \log \epsilon^5 n^2 F_2 G_2\right) + \frac{\partial}{\partial v}\log\frac{F_2}{G_2} = 0\ ;
$$

$$
(73) \quad M_1 = 2B_1, \quad \frac{\partial}{\partial u}\left(f + \log\frac{F_1 G_1}{\delta m^2}\right) + \delta\frac{\partial}{\partial v}\log\frac{F_1}{G_1} = \frac{\partial}{\partial u}\left(2f - \log \delta^2 m^4\right)\ ;
$$

---

* E. J. Wilczynski, *Brussels Paper*, p. 28 and p. 43.

(74) $M_2 = 2C_2,\quad \epsilon \dfrac{\partial}{\partial u} \log \dfrac{F_2}{G_2} + \dfrac{\partial}{\partial v}\left(f + \log \dfrac{F_2G_2}{\epsilon n^2}\right) = \dfrac{\partial}{\partial v}\left(2f - \log \epsilon^2 n^4\right)$;

(75) $S_1 = 2C_2,\quad \dfrac{1}{\delta} \dfrac{\partial}{\partial u} \log \dfrac{F_1}{G_1} + \dfrac{\partial}{\partial v}\left(3f + \log \dfrac{\delta^7 m^2}{n^4} F_1G_1\right) = \dfrac{\partial}{\partial v}\left(2f - \log \epsilon^2 n^4\right)$;

(76) $S_2 = 2B_1,\quad \dfrac{\partial}{\partial u}\left(3f + \log \dfrac{\epsilon^7 n^2}{m^4} F_2G_2\right) + \dfrac{1}{\epsilon} \dfrac{\partial}{\partial v} \log \dfrac{F_2}{G_2} = \dfrac{\partial}{\partial u}\left(2f - \log \delta^2 m^4\right).$

If we multiply (75) by δ and subtract (71) from the result, we find

$$\frac{\partial}{\partial v} \log \delta^2\epsilon^2 = 0 .$$

In a similar fashion from (72) and (76) one obtains

$$\frac{\partial}{\partial u} \log \delta^2\epsilon^2 = 0 ,$$

so that

(77) $\delta\epsilon = k$, where k is an arbitrary constant.

Using this relation, the six conditions reduce to the four given by (71), (72) and the two following:

(78) $$\frac{\partial}{\partial u}\left(f - \log \delta m^2 F_1G_1\right) - \delta \frac{\partial}{\partial v} \log \frac{F_1}{G_1} = 0 ,$$

(79) $$\frac{\partial}{\partial u} \log \frac{F_2}{G_2} - \frac{1}{\epsilon} \frac{\partial}{\partial v}\left(f - \log \epsilon n^2 F_2G_2\right) = 0 .$$

Adding (71) and (78) gives, using (11),

(80) $$G_{1u} + 2\delta F_1G_1 - \delta_v G_1 - \delta G_{1v} = 0,$$

while subtracting (78) from (71) gives

(81) $$F_{1u} - 2\delta F_1G_1 + \delta_v F_1 + \delta F_{1v} = 0.$$

Adding (80) and (81) gives

$$\frac{\partial}{\partial u}\left(F_1 + G_1\right) + (F_1 - G_1)\,\delta_v + \delta \frac{\partial}{\partial v}\left(F_1 - G_1\right) = 0,$$

which, since $F_1 + G_1 = -2C_1$, $F_1 - G_1 = -\frac{2}{\delta}B_1$, becomes

$$C_{1u} + B_{1v} = 0.$$

But from (11),

$$C_{1u} + B_{1v} = \frac{\partial^2}{\partial u \partial v}\left(f + \log \delta\right),$$

so that

$$\frac{\partial^2}{\partial u \partial v}\left(f + \log \delta\right) = 0. \tag{82}$$

By a similar calculation, from (72) and (79), one obtains

$$\frac{\partial^2}{\partial u \partial v}\left(f + \log \epsilon\right) = 0. \tag{83}$$

From (82), (83) and (77) it follows that $f_{uv} = 0$, whence

$$k^2 = \delta^2\epsilon^2 = \frac{c'd}{mn} = 1, \quad \text{and } k = \pm 1.$$

But $\epsilon = \pm 1/\delta$ enters in such a way into equations (71), (72), (78) and (79) that the same set of equations results from using $k = -1$ as from using $k = 1$. So there is no loss of generality in assuming $k = 1$. Use the transformation group (3) to make

$$f = \text{const.}, \quad c = d' = 0\ ; \tag{84}$$

these relations will be left unaltered by the sub-group of (3) given by

$$\lambda = \frac{c_1}{\sqrt{\alpha_u}}, \quad \mu = \frac{c_2}{\sqrt{\beta_v}},$$

where c_1 and c_2 are arbitrary constants.

From (84) and (82), it follows that

$$\delta^2 = -\frac{d}{m} = -\varphi(u)\psi(v)\ ;$$

then by using transformations (3) we can choose α and β so as to make

$$d = m, \tag{85}$$

and thereafter use the sub-group of (3) given by

$$\lambda = \frac{c_1}{c_2}, \quad \mu = \frac{c_2}{c_3}, \quad \alpha_u = \beta_v = C_3^2$$

where c_1, c_2 and c_3 are arbitrary constants.

Since $mn - c'd = f_{uv} = 0$, we find further

$$c' = n. \tag{86}$$

These equations show that the developables of the congruence intercept isothermally conjugate nets on both sheets of the focal surface.*

Subtracting (80) from (81), and using (11) and (85), one finds

$$m_{uu} + m_{vv} = 0. \tag{87}$$

By a similar computation based on the two equations corresponding to (80) and (81) one obtains

$$n_{uu} + n_{vv} = 0. \tag{88}$$

The integrability conditions become

$$b = -m_v, \quad a' = -n_u, \quad b' = -a, \tag{89}$$

$$a_v = 2\frac{\partial}{\partial u} mn, \quad a_u = -2\frac{\partial}{\partial v} mn. \tag{90}$$

If we differentiate the second equation of (90) with respect to v, and the first equation with respect to u, and subtract one resulting equation from the other, we obtain

$$(mn)_{uu} + (mn)_{vv} = 0. \tag{91}$$

If we perform the differentiations indicated in (91), and use (87) and (88), we obtain

$$\frac{m_u}{n_v} = -\frac{m_v}{n_u} = \rho, \tag{92}$$

* E. J. Wilczynski, *General theory of congruences*, these Transactions, vol. 16 (1915), p. 322.

where ρ is a function of u and v which we may assume to be real, if the focal sheets of the congruence are real. From (92) we find

$$m_{uu} = \rho_u n_v + \rho n_{uv}, \quad m_{vv} = - \rho_v n_u - \rho n_{uv} ,$$

whence, by (87),

(93) $$\rho_u n_v - \rho_v n_u = 0.$$

Again we find

$$m_{uv} = \rho_v n_v + \rho n_{vv} = - \rho_u n_u - \rho n_{uu},$$

whence, by (88),

(94) $$\rho_u n_u + \rho_v n_v = 0.$$

Equations (93) and (94) are two homogeneous equations which n_u and n_v must satisfy. The determinant of the coefficients must be zero;* i.e., $\rho_u^2 + \rho_v^2 = 0$, whence $\rho_u = \rho_v = 0$ since ρ is real, and $\rho = k$, where k is an arbitrary real constant. Then $k = -m_v/n_u = -b/a'$, using (2); so by using (3) we can make $k = 1$ ($k = 0$ gives the coincidence congruences) and thereafter use the subgroup of (3) for which

$$\lambda = \mu = c_1, \quad \alpha_u = \beta_v = c_2.$$

Equation (92) becomes

(95) $$m_u = - n_v, \quad m_v = n_u;$$

using these values, equations (90) can be integrated and give

(96) $$a = n^2 - m^2 + k,$$

where k is an arbitrary constant.

The congruences which possess the coincidences $(d_1'd_2'')$ and $(d_1''d_2')$ are identical with the congruences which arise from the Riemann sphere representation of functions of a complex vairable according to one of the methods discussed

* Or else $n_u = n_v = 0$; then $n =$ const. and if $n \neq 0$ we can make $n = 1$ and are led to the coincidence congruences, which form a special case of the general set of congruences possessing the given coincidences.

by Wilczynski.* We shall call them *bidirectrix* congruences since the four directrices associated in general with a line of the congruence in this case reduce to two. Thus *the property of possessing coincidences* $(d_1'd_2'')$ *and* $(d_1''d_2')$ *characterizes completely this set of bidirectrix congruences, discovered by Wilczynski.*

Using Wilczynski's results in connection with the theorem just stated we see that *a non-degenerate congruence in the real and distinct focal sheets, which possesses the coincidences* $(d_1'd_2'')$ *and* $(d_1''d_2')$, *has also the following properties*:

(*a*) *it is a W-congruence*;

(*b*) *its developables intercept isothermally conjugate nets on both sheets of the focal surface*;

(*c*) *the asymptotic curves on both focal sheets belong to linear complexes*;

(*d*) *corresponding asymptotic curves belong to the same complex*;†

(*e*) *the directrix curves of the two sheets of the focal surface correspond to each other*;

(*f*) *on each sheet of the focal surface, each of its axis curve tangents is conjugate to one of the ray curve tangents.*

Moreover *it has the canonical form*

$$c = d' = 0,\ d = m,\ c' = n,\ a' = -b = -n_u,\ a = -b' = n^2 - m^2 + k,$$

where

$$m_u = n_v, \quad m_v = -n_u$$

and k is an arbitrary constant.

* *Line geometric representations for functions of a complex variable*, these Transactions, vol. 20 (1919), pp. 283–298.

A set of properties characteristic of a class of congruences connected with the theory of functions, these Transactions, vol. 21 (1920), pp. 409–445.

† This property is added by the author and may be proved as follows. Denote by Γ the linear complex to which the asymptotic curve $v_1 = \text{const.}$ on S_y belongs so that the linear complex osculating $v_1 = \text{const.}$ at any point is Γ. In view of the coincidence property, the two linear complexes osculating $u_1 = \text{const.}$ and $v_1 = \text{const.}$ which pass through P_y have in common a linear congruence whose directrices are the directrices of the first kind of P_y and P_z. All the lines which meet these two directrices belong to Γ, so that Γ may be described as follows: consider the directrices of the first kind of the points on an asymptotic curve on one focal sheet, and the corresponding directrices of the first kind along the corresponding asymptotic curve on the other focal sheet; then the linear complex to which the first asymptotic curve belongs is made up of the totality of lines which meet a pair of corresponding directrices. Evidently the corresponding asymptotic curve on the other sheet also belongs to the same linear complex.

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ASYMPTOTIC PLANETOIDS*

BY

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1. **Introduction.** Near the vertices of the equilateral triangles described in the plane of Jupiter's orbit and on the line joining the Sun and Jupiter as base are to be found six of the planetoids. Four of these,† Achilles (588), Patrocles (617), Hector (624), and Nestor (689), oscillate about one vertex in much the same way as the mythological luminaries of the same names circulated about the walls of Troy. Two planetoids are found at the other vertex, but their names are unknown to the author.‡ These six planetoids are in the vicinity of two of the five well known points of libration in the problem of three bodies.§ The three other points lie on the line joining the Sun and Jupiter, one point lying between the Sun and Jupiter, another on the side of the sun remote from Jupiter, and the third on the side of Jupiter remote from the Sun. One of these straight-line points in the case of the Sun and Earth, viz., the one on the side of the Earth remote from the Sun, has a physical significance in that it may account for the "Gegenshein."|| The equilateral triangle points of libration were considered by Lagrange in his celebrated prize memoir of 1772 as "pure curiosities," but recent astronomical discoveries show that these points likewise have some physical significance attached to them.

The object of this article is to determine orbits for these planetoids of the Sun and Jupiter which will approach the equilateral triangle points of libration as the time approaches infinity. As these orbits are asymptotic to the above mentioned points of libration, we have designated the planetoids which move in such orbits as "asymptotic planetoids." These planetoids are considered to be of appreciable mass, and their perturbations upon the Sun and Jupiter are determined. They are assumed to move in the plane of Jupiter's orbit.

* Presented to the Society, December 31, 1919.

† Marcolongo, *Il problema dei tre corpi*, Scientia, vol. 1 (1919), No. 8.

‡ I am indebted to Professor E. W. Brown for this information, as he communicated it to me when this paper was presented to the Society.

§ Lagrange, *Collected Works*, vol. 6, pp. 229–324; Tisserand, *Mécanique Céleste*, vol. 1, chap. 8; Moulton, *Celestial Mechanics*, pp. 309–318.

|| Gyldèn, *Sur un Cas Particulier du Problème Astronomique*, vol. 1; Moulton, *Celestial Mechanics*, p. 305.

In plain mathematical language, divorced from every astronomical application, the paper treats of two-dimensional asymptotic orbits near the equilateral triangle equilibrium points in the problem of three finite bodies. The paper concludes with numerical examples of orbits in which the ratios of the masses are not those of the Sun, Jupiter and the planetoids, but are chosen so that the orbits near each point can be drawn to the same scale.

Several classes of asymptotic orbits have already been obtained, but with one exception, number (6) below, they belong to the particular case of the problem of three bodies in which one body is infinitesimal and the finite bodies move in circles. The following is a list of the solutions which have been obtained:

(1) Warren determined two-dimensional orbits which are asymptotic to the points of libration lying on the straight line joining the two finite bodies. (American Journal of Mathematics, vol. 38, No. 3, pp. 221–248.)

The remaining cases were found by the author of the present paper.

(2) Two- and three-dimensional orbits which are respectively asymptotic to the two- and three-dimensional periodic orbits near the straight line equilibrium points determined by Moulton* in the chapter on *Oscillating satellites*. (American Journal of Mathematics, vol. 41, No. 2, pp. 79–110.)

(3) Two-dimensional orbits which approach the equilateral triangle points of libration. (Transactions of the Cambridge Philosophical Society, vol. 22, No. 15, pp. 309–340.)

(4) Three-dimensional orbits which are asymptotic to the three-dimensional periodic oscillations near the equilateral triangle equilibrium points which were determined by Buck.† This paper forms part of the paper mentioned in (3).

(5) Three-dimensional orbits which are asymptotic to the isosceles triangle solutions‡ of the problem of three bodies. (Proceedings of the London Mathematical Society, ser. 2, vol. 17, No. 1, pp. 54–74.)

(6) Two-dimensional orbits which are asymptotic to the straight line equilibrium points when the three masses are finite.§

2. **The differential equations.** Let m_1, m_2, and m_3 represent the masses of the three bodies and let M denote their sum. Let a system of rectangular axes be chosen having the origin at the center of mass of the three bodies and the plane of their motion as the plane of reference. Let the axes rotate about the origin with the uniform angular velocity n and let the coördinates of the three bodies m_i be denoted by (x_i, y_i), $i = 1, 2, 3$. If the bodies are subject to their mutual

* *Periodic Orbits*, chap. 5.

† Buck, chap. 9 of Moulton's *Periodic Orbits*.

‡ Buchanan, chap. 10 of Moulton's *Periodic Orbits*.

§ Rendiconti del Circolo Matematico di Palermo, vol. 45, pp. 1–25.

attractions according to the Newtonian law, then the differential equations which define their motion are

$$
\begin{aligned}
&x_i'' - 2ny_i' - n^2 x_i = \frac{1}{m_i}\frac{\partial U}{\partial x_i}, \\
&y_i'' + 2nx_i' - n^2 y_i = \frac{1}{m_i}\frac{\partial U}{\partial y_i} \qquad (i = 1, 2, 3), \\
&U = k^2\left[\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}}\right], \\
&r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2} \qquad (i, j = 1, 2, 3;\ i \neq j),
\end{aligned} \tag{1}
$$

where k^2 is the factor of proportionality and the accents denote derivation with respect to t.

These equations admit the vis viva integral

$$\sum_{i=1}^{3} m_i(x_i'^2 + y_i'^2 - x_i^2 - y_i^2) = 2U + C, \tag{2}$$

where C is the constant of integration.

It is shown in treatises on celestial mechanics* that the equilateral triangle configuration with proper initial components of velocity is a particular solution of the differential equations of motion (1). If the units of distance and time are so chosen that the mutual distances and k^2 are unity, then the angular velocity of rotation n must satisfy the condition

$$n^2 = M.$$

Since the unit of mass is so far arbitrary, it is possible, without loss of generality, to put $M = 1$. Hence n^2 is also unity.

If the axes are initially orientated so that the x-axis is parallel to the side of the equilateral triangle joining m_2 and m_3, then the values of the coördinates for the particular solutions are either

Configuration I (Fig. I),

$$
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2), & y_1 &= \frac{1}{2}\sqrt{3}\,(m_2 + m_3), \\
x_2 &= \frac{1}{2}m_1 + m_3, & y_2 &= -\frac{1}{2}\sqrt{3}\,m_1, \\
x_3 &= -\left(\frac{1}{2}m_1 + m_2\right), & y_3 &= -\frac{1}{2}\sqrt{3}\,m_1,
\end{aligned} \tag{3}
$$

* See Moulton's *Introduction to Celestial Mechanics*, pp. 309–11.

or

Configuration II (Fig. 2),

$$(4) \qquad \begin{aligned} x_1 &= \frac{1}{2}(m_3 - m_2), & y_1 &= -\frac{1}{2}\sqrt{3}\,(m_2 + m_3), \\ x_2 &= \frac{1}{2}m_1 + m_3, & y_2 &= \frac{1}{2}\sqrt{3}\,m_1, \\ x_3 &= -\left(\frac{1}{2}m_1 + m_2\right), & y_3 &= \frac{1}{2}\sqrt{3}\,m_1. \end{aligned}$$

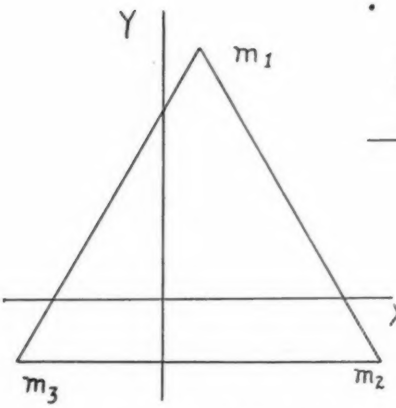

FIG. 1.—Configuration I.

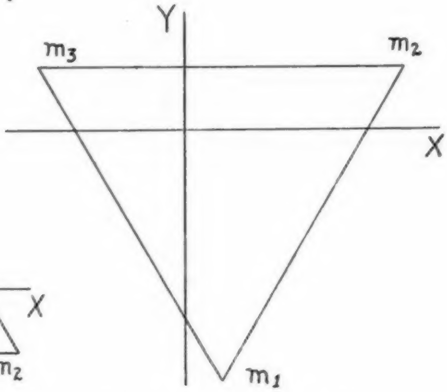

FIG. 2.—Configuration II.

Since the origin of coördinates is at the center of mass, it follows that

$$m_1x_1 + m_2x_2 + m_3x_3 = 0, \qquad m_1y_1 + m_2y_2 + m_3y_3 = 0,$$

and therefore the coördinates of one of the bodies may be eliminated from the differential equations. Let us suppose x_3 and y_3 are thus eliminated by

$$(5) \qquad x_3 = -\frac{1}{m_3}(m_1x_1 + m_2x_2), \qquad y_3 = -\frac{1}{m_3}(m_1y_1 + m_2y_2).$$

When these substitutions are made in (1), and k^2 and n are both put equal to unity, the differential equations of motion become

$$
(6)\quad
\begin{aligned}
x_1'' - 2y_1' - x_1 &= -\frac{m_2(x_1 - x_2)}{\rho_{12}^3} - \frac{(m_1 + m_3)x_1 + m_2 x_2}{\rho_{13}^3},\\
y_1'' + 2x_1' - y_1 &= -\frac{m_2(y_1 - y_2)}{\rho_{12}^3} - \frac{(m_1 + m_3)y_1 + m_2 y_2}{\rho_{13}^3},\\
x_2'' - 2y_2' - x_2 &= -\frac{m_1(x_2 - x_1)}{\rho_{12}^3} - \frac{m_1 x_1 + (m_2 + m_3)x_2}{\rho_{23}^3},\\
y_2'' + 2x_2' - y_2 &= -\frac{m_1(y_2 - y_1)}{\rho_{12}^3} - \frac{m_1 y_1 + (m_2 + m_3)y_2}{\rho_{23}^3},\\
\rho_{12} &= [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2},\\
\rho_{13} &= \frac{1}{m_3}[\{(1 - m_2)x_1 + m_2 x_2\}^2 + \{(1 - m_2)y_1 + m_2 y_2\}^2]^{1/2},\\
\rho_{23} &= \frac{1}{m_3}[\{m_1 x_1 + (1 - m_1)x_2\}^2 + \{m_1 y_1 + (1 - m_1)y_2\}^2]^{1/2}.
\end{aligned}
$$

3. **The equations of variations.** Let

$$
(7)\quad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + \epsilon u_1, & y_1 &= \frac{1}{2}\sqrt{3}\,(m_2 + m_3) + \epsilon v_1,\\
x_2 &= \frac{1}{2}m_1 + m_3 + \epsilon u_2, & y_2 &= -\frac{1}{2}\sqrt{3}\,m_1 + \epsilon v_2,
\end{aligned}
$$

where $u_1, \ldots, v_2$ are new dependent variables, and ϵ is an arbitrary parameter. The additive quantities $\epsilon u_1, \ldots, \epsilon v_2$ denote the components of displacement of m_1 and m_2 from the vertices of the equilateral triangle in configuration I (Fig. 1). We shall not consider configuration II (Fig. 2) in detail as the coördinates in Fig. 2 differ from the corresponding ones in Fig. 1 only in the sign of $\sqrt{3}$. Since $\sqrt{3}$ does not occur explicitly in the differential equations, we may change the sign of $\sqrt{3}$ in any solutions which we obtain for configuration I and thereby obtain the corresponding solutions for configuration II.

When equations (7) are substituted in (6) the right members of the differential equations can be expanded as power series in ϵ which will converge for ϵ sufficiently small. Since the expressions in (7) that are independent of ϵ are particular solutions of equations (6), there will be no terms independent of ϵ after the substitutions (7) have been made. Hence the factor ϵ can be divided out of the equations, and when m_3 is replaced by $1 - m_1 - m_2$ the differential equations (6) become

$$
(8)\quad
\begin{aligned}
&\left(D_2 - \frac{3}{4}\right)u_1 - \left\{ 2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_1 + 0u_2 - \frac{3\sqrt{3}}{2} m_2 v_2 \\
&\qquad = \epsilon U_1^{(2)} + \epsilon^2 U_1^{(3)} + \cdots + \epsilon^{n-1} U_1^{(n)} + \cdots, \\
&\left\{2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_1 + \left(D^2 - \frac{9}{4}\right) v_1 - \frac{3\sqrt{3}}{2} m_2 u_2 + 0v_2 \\
&\qquad = \sqrt{3}\,[\epsilon V_1^{(2)} + \epsilon^2 V_1^{(3)} + \cdots + \epsilon^{n-1} V_1^{(n)} + \cdots], \\
&-\frac{9}{4} m_1 u_1 - \frac{3\sqrt{3}}{4} m_1 v_1 + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_2 - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_2 \\
&\qquad = \epsilon U_2^{(2)} + \epsilon^2 U_2^{(3)} + \cdots + \epsilon^{n-1} U_2^{(n)} + \cdots, \\
&-\frac{3\sqrt{3}}{4} m_1 u_1 + \frac{9}{4} m_1 v_1 + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_2 + \left(D^2 - \frac{9}{4} m_1\right) v_2 \\
&\qquad = \sqrt{3}\,[\epsilon V_2^{(2)} + \epsilon^2 V_2^{(3)} + \cdots + \epsilon^{n-1} V_2^{(n)} + \cdots],
\end{aligned}
$$

where D denotes the operator d/dt, and $U_1^{(k)}, \ldots, V_2^{(k)}$ $(k = 2, \ldots, n, \ldots)$ are homogeneous polynomials in u_1, u_2, v_1, v_2 of degree k. In all these polynomials, $\sqrt{3}$ occurs only as a factor of the odd powers of v_1 and v_2, considered together.

If we consider only the linear terms in (8) we obtain the equations of variation. They are

$$
(9)\quad
\begin{aligned}
&\left(D^2 - \frac{3}{4}\right)u_1 - \left\{2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_1 + 0u_2 - \frac{3\sqrt{3}}{2} v_2 = 0, \\
&\left\{2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_1 + \left(D^2 - \frac{9}{4}\right) v_1 - \frac{3\sqrt{3}}{2} m_2 u_2 + 0v_2 = 0, \\
&-\frac{9}{4} m_1 u_1 - \frac{3\sqrt{3}}{4} m_1 v_1 + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_2 - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_2 = 0, \\
&-\frac{3\sqrt{3}}{4} m_1 u_1 + \frac{9}{4} m_1 v_1 + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_2 + \left(D^2 - \frac{9}{4} m_1\right) v_2 = 0.
\end{aligned}
$$

4. **The solutions of the equations of variations.** The solutions of the equations of variation can usually be obtained by differentiating the generating solutions, equations (3), with respect to the arbitrary constants which denote the initial time and the scale factor.* This is not possible in the case under consideration since the generating solutions are constants.

Equations (9) are linear differential equations with constant coefficients, and in order to obtain their solutions we consider the operator D as an alge-

* Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, chap. 4.

braic quantity and equate to zero the determinant formed from the coefficients of u_1, u_2, v_1, v_2 in (9). This gives

$$(10)\quad \Delta = \begin{vmatrix} D^2 - \frac{3}{4}, & -2D - \frac{3\sqrt{3}}{4}(1-2m_2), & 0, & -\frac{3\sqrt{3}}{2}m_2 \\ 2D - \frac{3\sqrt{3}}{4}(1-2m_2), & D^2 - \frac{9}{4}, & -\frac{3\sqrt{3}}{2}m_2, & 0 \\ -\frac{9}{4}m_1, & -\frac{3\sqrt{3}}{4}m_1, & D^2 - 3 + \frac{9}{4}m_1, & -2D + \frac{3\sqrt{3}}{4}m_1 \\ -\frac{3\sqrt{3}}{4}m_1, & \frac{9}{4}m_1, & 2D + \frac{3\sqrt{3}}{4}m_1, & D^2 - \frac{9}{4}m_1 \end{vmatrix} = 0.$$

This determinant reduces to

$$(11)\quad \Delta = D^2\left[D^6 + 2D^4 + \left\{1 + \frac{27}{4}(m_1m_2 + m_2m_3 + m_3m_1)\right\}D^2 + \frac{27}{4}(m_1m_2 + m_2m_3 + m_3m_1)\right] = 0.$$

If we neglect the factor D^2, the preceding equation is a cubic in D^2 and its discriminant is

$$\frac{27}{16}(m_1m_2 + m_2m_3 + m_3m_1)^2\,[27(m_1m_2 + m_2m_3 + m_3m_1) - 1].$$

Since $m_1 + m_2 + m_3 = 1$ and since m_1, m_2, and m_3 are all positive, this discriminant is positive, and therefore the cubic has one negative root and two conjugate complex roots. Let these roots be λ_1^2, λ_2^2, and λ_3^2 where

$$(12)\quad \lambda_1 = \sqrt{-1}\,\lambda, \qquad \lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu.$$

Then the solutions of (11) are

$$(13)\quad D = 0, \quad 0, \quad \pm\lambda_1, \quad \pm\lambda_2, \quad \pm\lambda_3,$$

and the solutions of the differential equations (9) are therefore

$$(14)\quad \begin{aligned} u_1 &= \alpha_1 + \alpha_2 t + \alpha_3 e^{\lambda_1 t} + \alpha_4 e^{-\lambda_1 t} + \alpha_5 e^{\lambda_2 t} + \alpha_6 e^{-\lambda_2 t} + \alpha_7 e^{\lambda_3 t} + \alpha_8 e^{-\lambda_3 t}, \\ v_1 &= \beta_1^{(0)}\alpha_1 + \beta_2^{(0)}\alpha_2 t + \beta_1^{(1)}\alpha_3 e^{\lambda_1 t} - \beta_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \beta_1^{(2)}\alpha_5 e^{\lambda_2 t} - \beta_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \beta_1^{(3)}\alpha_7 e^{\lambda_3 t} - \beta_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \\ u_2 &= \gamma_1^{(0)}\alpha_1 + \gamma_2^{(0)}\alpha_2 t + \gamma_1^{(1)}\alpha_3 e^{\lambda_1 t} + \gamma_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \gamma_1^{(2)}\alpha_5 e^{\lambda_2 t} + \gamma_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \gamma_1^{(3)}\alpha_7 e^{\lambda_3 t} + \gamma_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \\ v_2 &= \delta_1^{(0)}\alpha_1 + \delta_2^{(0)}\alpha_2 t + \delta_1^{(1)}\alpha_3 e^{\lambda_1 t} - \delta_2^{(1)}\alpha_4 e^{-\lambda_1 t} + \delta_1^{(2)}\alpha_5 e^{\lambda_2 t} - \delta_2^{(2)}\alpha_6 e^{-\lambda_2 t} \\ &\quad + \delta_1^{(3)}\alpha_7 e^{\lambda_3 t} - \delta_2^{(3)}\alpha_8 e^{-\lambda_3 t}, \end{aligned}$$

where $\alpha_1, \ldots, \alpha_8$ are the constants of integration. The remaining symbols $\beta_1^{(0)}, \ldots, \delta_2^{(3)}$ denote constants which are defined by the following equations and properties:

$$\beta_1^{(0)} = \frac{1}{1-m_1}\left[\frac{8}{9} - \frac{\sqrt{3}}{3}(1-m_1) - \frac{2}{3}(m_1 - \sqrt{3}\, m_2) + \frac{2}{9(m_1-1)}\{m_1(m_2+m_3) + 4m_2m_3\}\right],$$

$$\gamma_1^{(0)} = \frac{(m_1-1)(\sqrt{3}\, m_1 + 9m_1m_2) + \sqrt{3}\,\{m_1(m_2+m_3)+4m_2m_3\}}{9m_2(m_1-1)^2},$$

$$\delta_1^{(0)} = -\frac{1}{2m_2}\left[(1-2m_2)\,\beta_1^{(0)} + \frac{8(m_3-m_2)}{9(m_1-1)} + \frac{\sqrt{3}}{3}\right],$$

$$\beta_2^{(0)} = \frac{m_3-m_2}{\sqrt{3}(m_1-1)}, \qquad \gamma_2^{(0)} = \frac{m_1}{m_1-1}, \qquad \delta_2^{(0)} = \frac{m_1+2m_3}{\sqrt{3}(m_1-1)},$$

$$\beta_1^{(i)} = \frac{3\sqrt{3}\,[(m_3-m_2)\lambda_i^2 + 2\sqrt{3}(1-m_1)\lambda_i + 3(m_2-m_3)]}{4\lambda_i^4 - \lambda_i^2(5-9m_1) + 6\sqrt{3}(m_3-m_2)\lambda_i + 27(1-m_1)},$$

$$\gamma_1^{(i)} = \frac{2\sqrt{3}}{9m_2}\left(\lambda_i^2 - \frac{9}{4}\right)\beta_1^{(i)} + \frac{4\sqrt{3}}{9m_2}\lambda_i - \frac{1-2m_2}{2m_2},$$

$$\delta_1^{(i)} = \frac{2\sqrt{3}}{9m_2}\left(\lambda_i^2 - \frac{3}{4}\right) - \left\{\frac{4\sqrt{3}}{9m_2}\lambda_i + \frac{1-2m_2}{2m_2}\right\}\beta_1^{(i)} \qquad (i = 1, 2, 3).$$

The following pairs of constants differ only in the sign of $\sqrt{3}$:

$$\beta_1^{(i)}, \beta_2^{(i)}; \quad \gamma_1^{(i)}, \gamma_2^{(i)}; \quad \delta_1^{(i)}, \delta_2^{(i)}; \qquad (i = 1, 2, 3).$$

The following pairs are conjugate complex:

$$\beta_1^{(1)}, -\beta_2^{(1)}; \quad \gamma_1^{(1)}, \gamma_2^{(1)}; \quad \delta_1^{(1)}, -\delta_2^{(1)}; \quad \beta_k^{(2)}, \beta_k^{(3)}; \quad \gamma_k^{(2)}, \gamma_k^{(3)}; \quad \delta_k^{(2)}, \delta_k^{(3)}; \quad (k = 1, 2).$$

The solutions for D in equations (13) are called the *characteristic exponents*. Since the original differential equations (1) admit the integral (2), then two of the characteristic exponents will be zero.* It was to be expected, therefore, that two roots of (11) should be zero. Poincaré† has also shown that if equa-

* Poincaré, loc. cit., p. 188.

† *Loc. cit.*, p. 69.

tions (1) do not contain t explicitly, which is always the case in all problems of mechanics in which there is a conservative system, then the characteristic exponents are always equal in pairs but opposite in sign. It is this property of the solutions that makes the determinant (11) even in D.

5. **Construction of Asymptotic Solutions in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$.** We shall now construct solutions of the differential equations (6) which are asymptotic in the sense of Poincaré,* that is, *each* term of the solution must have the form

$$e^{ct}\,P(t),$$

where c is a constant having its real part different from zero, and $P(t)$ is a periodic function of t or, in particular, a constant. Such solutions will therefore approach zero as t approaches $+\infty$ or $-\infty$ according as the real part of c is negative or positive, respectively.

The only terms of the solutions of the equations of variation which are asymptotic in the sense just defined are those in $e^{\pm\lambda_2 t}$ and $e^{\pm\lambda_3 t}$. These terms approach zero as t approaches $-\infty$ or $+\infty$ according as the $+$ or $-$ signs are taken with λ_2 and λ_3.

In this section we shall construct the solutions which approach zero as t approaches $+\infty$. These solutions will obviously involve powers of $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. In the next section we shall show how the solutions in $e^{+\lambda_2 t}$ and $e^{+\lambda_3 t}$ can be obtained directly from the solutions in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$.

Only the formal construction of the solutions is considered in §§ 5 and 6, but the convergence of the solutions obtained is established in § 7.

We propose to integrate equations (8) as power series in ϵ and this is the reason why ϵ was introduced in equations (7). Accordingly we substitute

$$(15)\qquad u_i = \sum_{j=0}^{\infty} u_{ij}\,\epsilon^j, \qquad v_i = \sum_{j=0}^{\infty} v_{ij}\,\epsilon^j \qquad (i = 1, 2),$$

in equations (8), and as we shall have frequent occasion to refer to the resulting equations we shall cite them as (8′). These equations (8′) are to be satisfied identically in ϵ and we may therefore equate the coefficients of the same powers of ϵ. In this way we obtain sequences of differential equations in u_{ij} and v_{ij} which can be integrated step by step, as we shall show, subject to suitable initial conditions and restrictions.

The first two steps of the integration will be considered in detail, and then an induction to the general term will be made to show that the process of integration can be carried on indefinitely.

---

* Loc. cit., p. 340.

Step 1. *Terms in* (8′) *independent of* ϵ. The terms in (8′) which are independent of ϵ are obviously the same as the equations of variation (9) if the subscripts on the dependent variables in (9) are altered so as to read u_{10}, v_{10}, u_{20} and v_{20} instead of u_1, u_2, v_1, v_2. Hence the solutions of these equations are the same as (14) with the corresponding changes of subscripts. Now only two of the exponentials, viz., $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$, approach zero as t approaches $+\infty$ and we therefore put equal to zero the arbitrary constants associated with the other exponentials. The solutions at this step are, then,

$$
(16)\quad
\begin{aligned}
u_{10} &= \alpha_6^{(0)} e^{-\lambda_2 t} + \alpha_8^{(0)} e^{-\lambda_3 t}, & v_{10} &= -\beta_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} - \beta_2^{(3)} \alpha_8^{(0)} e^{-\lambda_3 t},\\
u_{20} &= \gamma_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} + \gamma_2^{(3)} \alpha_8^{(0)} e^{-\lambda_3 t}, & v_{20} &= -\delta_2^{(2)} \alpha_6^{(0)} e^{-\lambda_2 t} - \delta_2^{(3)} \alpha_8^{(0)} e^{-\lambda_3 t},
\end{aligned}
$$

where $\alpha_6^{(0)}$ and $\alpha_8^{(0)}$ are arbitrary, and $\beta_2^{(2)}, \ldots, \delta_2^{(3)}$ are defined as in equations (14).

It is evident that the complementary functions at all the succeeding steps of the integrations will be the same as (14) and that after the exponentials are rejected which do not have the proper form, two constants of integration will remain undetermined. Hence, besides rejecting all the exponentials except $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$, it is necessary to impose two initial conditions upon these solutions so as to determine the arbitrary constants arising at each step of the integration.

Let us suppose that

$$
(17)\qquad u_1(0) = \epsilon_1, \qquad u_2(0) = \gamma,
$$

where ϵ_1 and γ are arbitrary. Since u_1 and u_2 are multiplied by the arbitrary parameter ϵ in (7) we may put ϵ_1 or γ equal to unity. Let us suppose $\epsilon_1 = 1$. When the conditions (17) are imposed on (15) it follows that

$$
(18)\quad
\begin{aligned}
u_{10}(0) &= 1, & u_{1j}(0) &= 0,\\
u_{20}(0) &= \gamma, & u_{2j}(0) &= 0 \qquad (j = 1, \ldots, \infty).
\end{aligned}
$$

Now applying these conditions to the solutions (16) we obtain

$$
\alpha_6^{(0)} = \frac{\gamma_2^{(3)} - \gamma}{\gamma_2^{(3)} - \gamma_2^{(2)}}, \qquad \alpha_8^{(0)} = \frac{\gamma - \gamma_2^{(2)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}.
$$

Since $\gamma_2^{(2)}$ and $\gamma_2^{(3)}$ are conjugate imaginaries, it follows that $\alpha_6^{(0)}$ and $\alpha_8^{(0)}$ are likewise conjugates.

In order to unify the notation, we put

$$
(19)\quad
\begin{aligned}
\alpha_6^{(0)} &= \alpha_{10}^{(10)}, & -\beta_2^{(2)} \alpha_6^{(0)} &= \beta_{10}^{(10)}, & \gamma_2^{(2)} \alpha_6^{(0)} &= \alpha_{20}^{(10)}, & -\delta_2^{(2)} \alpha_6^{(0)} &= \beta_{20}^{(10)},\\
\alpha_8^{(0)} &= \alpha_{10}^{(01)}, & -\beta_2^{(3)} \alpha_8^{(0)} &= \beta_{10}^{(01)}, & \gamma_2^{(3)} \alpha_8^{(0)} &= \alpha_{20}^{(01)}, & -\delta_2^{(3)} \alpha_8^{(0)} &= \beta_{20}^{(01)}.
\end{aligned}
$$

Since $\alpha_6^{(0)}$, $\alpha_8^{(0)}$; $\beta_2^{(2)}$, $\beta_2^{(3)}$; $\gamma_2^{(2)}$, $\gamma_2^{(3)}$; $\delta_2^{(2)}$, $\delta_2^{(3)}$ are conjugate pairs, it follows that the symbols in (19) which differ only in a permutation of superscripts are likewise conjugate pairs. This notation will be adopted in the sequel to denote conjugate complexes. Thus $\alpha_{i1}^{(jk)}$ and $\alpha_{i1}^{(jk)}$ $(j \neq k)$, are conjugate imaginaries.

On employing the above notation, we obtain for the desired solutions at this step

$$(20)\quad u_{i0} = \alpha_{i0}^{(10)} e^{-\lambda_2 t} + \alpha_{i0}^{(01)} e^{-\lambda_3 t}, \quad v_{i0} = \beta_{i0}^{(10)} e^{-\lambda_2 t} + \beta_{i0}^{(01)} e^{-\lambda_3 t} \qquad (i = 1, 2),$$

where $\alpha_{i0}^{(10)}, \ldots, \beta_{i0}^{(01)}$ are linear in γ.

If we put

$$\lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu,$$

and suppose that

$$\alpha_{i0}^{(10)} = \tfrac{1}{2}\,[a_{i0}^{(1)} + \sqrt{-1}\,b_{i0}^{(1)}], \qquad \beta_{i0}^{(10)} = \tfrac{1}{2}\,[c_{i0}^{(1)} + \sqrt{-1}\,d_{i0}^{(1)}],$$

then the solutions (20) become

$$(21)\quad \begin{aligned} u_{i0} &= e^{-\mu t}\,[a_{i0}^{(1)} \cos \nu t + b_{i0}^{(1)} \sin \nu t], \\ v_{i0} &= e^{-\mu t}\,[c_{i0}^{(1)} \cos \nu t + d_{i0}^{(1)} \sin \nu t]. \end{aligned}$$

This second form of the solutions is more convenient than (20) for numerical computation but is more cumbersome in obtaining the solutions at the succeeding steps.

Step 2. Coefficients of ϵ to the first degree. When the solutions (20) have been substituted in (8′), the differential equations obtained by equating the coefficients of ϵ to the first degree are

$$\begin{aligned}
&\left(D^2 - \frac{3}{4}\right) u_{11} - \left\{2D + \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} v_{11} + 0\,u_{21} - \frac{3\sqrt{3}}{2} m_2 v_{21} = U_{11}, \\
&\left\{2D - \frac{3\sqrt{3}}{4}(1 - 2m_2)\right\} u_{11} + \left(D^2 - \frac{9}{4}\right) v_{11} - \frac{3\sqrt{3}}{2} m_2 u_{21} + 0\,v_{21} = V_{11}, \\
(22)\quad &-\frac{9}{4} m_1 u_{11} - \frac{3\sqrt{3}}{4} m_1 v_{11} + \left(D^2 - 3 + \frac{9}{4} m_1\right) u_{21} \\
&\qquad\qquad - \left(2D - \frac{3\sqrt{3}}{4} m_1\right) v_{21} = U_{21}, \\
&-\frac{3\sqrt{3}}{4} m_1 u_{11} + \frac{9}{4} m_1 v_{11} + \left(2D + \frac{3\sqrt{3}}{4} m_1\right) u_{21} + \left(D^2 - \frac{9}{4} m_1\right) v_{21} = V_{21};
\end{aligned}$$

$$(23)\qquad \begin{aligned} U_{11} &= A_{11}^{(20)}\, e^{-2\lambda_2 t} + A_{11}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + A_{11}^{(02)}\, e^{-2\lambda_3 t}, \\ V_{11} &= B_{11}^{(20)}\, e^{-2\lambda_2 t} + B_{11}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + B_{11}^{(02)}\, e^{-2\lambda_3 t}, \\ U_{21} &= A_{21}^{(20)}\, e^{-2\lambda_2 t} + A_{21}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + A_{21}^{(02)}\, e^{-2\lambda_3 t}, \\ V_{21} &= B_{21}^{(20)}\, e^{-2\lambda_2 t} + B_{21}^{(11)}\, e^{-(\lambda_2+\lambda_3)t} + B_{21}^{(02)}\, e^{-2\lambda_3 t}, \end{aligned}$$

where $A_{11}^{(20)}, \ldots, B_{21}^{(02)}$ are quadratic expressions in γ having constant coefficients The constants like $A_{11}^{(20)}$ and $A_{11}^{(02)}$ which have the same subscripts but which have their superscripts reversed are conjugate imaginaries. The other constants $A_{11}^{(11)}, \ldots, B_{21}^{(11)}$ are real.

The complementary functions of equations (22) are the same as (14), and the particular integrals can be found by the method of the variation of parameters as at the previous step. These particular integrals are

$$(24)\qquad \begin{aligned} u_{i1} &= \sum_{j,\,k=0}^{2} \alpha_{i1}^{(jk)}\, e^{-(j\lambda_2+k\lambda_3)t} \qquad (j+k=2, \quad i=1,2), \\ v_{i1} &= \sum_{j,\,k=0}^{2} \beta_{i1}^{(jk)}\, e^{-(j\lambda_2+k\lambda_3)t}, \end{aligned}$$

where

$$(25)\qquad \begin{aligned} \alpha_{11}^{(jk)} &= \frac{\Delta_1^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}, & \beta_{11}^{(jk)} &= \frac{\Delta_2^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}, \\ \alpha_{21}^{(jk)} &= \frac{\Delta_3^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}, & \beta_{21}^{(jk)} &= \frac{\Delta_4^{(jk)}}{\Delta[-(j\lambda_2+k\lambda_3)]}. \end{aligned}$$

The preceding symbols are defined as follows:

$\Delta[-(j\lambda_2+k\lambda_3)]$ denotes the determinant Δ in (10) when D has been replaced by $-(j\lambda_2+k\lambda_3)$;

$\Delta_i^{(jk)}$, $i = 1, 2, 3, 4$, denotes the preceding determinant $\Delta[-(j\lambda_2+k\lambda_3)]$ when the elements of the ith row, reading from top to bottom, have been replaced by $A_{11}^{(jk)}, B_{11}^{(jk)}, A_{21}^{(jk)}, B_{21}^{(jk)}$, respectively.

The constants in (25), viz., $\alpha_{11}^{(jk)}, \ldots, \beta_{21}^{(jk)}$ are found to have the same form as the constants in (23). Thus $\alpha_{11}^{(jk)}$ and $\alpha_{11}^{(kj)}$ are conjugate complex if $j \neq k$, but real if $j = k$.

If the right members in (22) contained any constant terms, or terms in t or in $e^{\pm\lambda_i t}$, $i = 1, 2, 3$, that is, terms which are the same functions of t as the terms in the complementary functions, then the denominators in (25) would vanish and the particular integrals would not take the form (24), but would contain terms in t, t^2 or $t\, e^{\pm\lambda_i t}$. Hence, such terms would appear in the particular integrals if and only if the right members contained constants, terms in t or $e^{\pm\lambda_i t}$, respectively. In order, then, to show that there are no terms in the particular integrals which do not satisfy Poincaré's definition of asymptotic solutions, it

is sufficient to show that the right members do not contain any terms which are the same functions of t as any part of the complementary functions.

To obtain the complete solutions of (22), we combine the complementary functions, equations similar to (14), and the particular integrals (24). The particular integrals have the desired form for asymptotic solutions, but we must reject all the terms of the complementary functions except those in $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. This is possible, of course, by equating to zero all the arbitrary constants of integration except those associated with the exponentials $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$. The desired solutions are therefore

$$
\begin{aligned}
u_{11} &= \alpha_6^{(1)} e^{-\lambda_2 t} + \alpha_8^{(1)} e^{-\lambda_3 t} + \alpha_{11}^{(20)} e^{-2\lambda_2 t} + \alpha_{11}^{(11)} e^{-(\lambda_2+\lambda_3)t} + \alpha_{11}^{(02)} e^{-2\lambda_3 t},\\
v_{11} &= -\beta_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} - \beta_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \beta_{11}^{(20)} e^{-2\lambda_2 t} + \beta_{11}^{(11)} e^{-(\lambda_2+\lambda_3)t}\\
&\qquad + \beta_{11}^{(02)} e^{-2\lambda_3 t},\\
u_{21} &= \gamma_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} + \gamma_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \alpha_{21}^{(20)} e^{-2\lambda_2 t} + \alpha_{21}^{(11)} e^{-(\lambda_2+\lambda_3)t} + \alpha_{21}^{(02)} e^{-2\lambda_3 t},\\
v_{21} &= -\delta_2^{(2)} \alpha_6^{(1)} e^{-\lambda_2 t} - \delta_2^{(3)} \alpha_8^{(1)} e^{-\lambda_3 t} + \beta_{21}^{(20)} e^{-2\lambda_2 t} + \beta_{21}^{(11)} e^{-(\lambda_2+\lambda_3)t}\\
&\qquad + \beta_{21}^{(02)} e^{2\lambda_3 t},
\end{aligned}
$$

where $\alpha_6^{(1)}$ and $\alpha_8^{(1)}$ are the constants of integration. By virtue of the initial conditions (18), these constants must satisfy the equations

$$
\begin{aligned}
\alpha_6^{(1)} + \alpha_8^{(1)} &= -\,[\alpha_{11}^{(20)} + \alpha_{11}^{(11)} + \alpha_{11}^{(02)}] \equiv R_1^{(1)},\\
\gamma_2^{(2)} \alpha_6^{(1)} + \gamma_2^{(3)} \alpha_8^{(1)} &= -\,[\alpha_{21}^{(20)} + \alpha_{21}^{(11)} + \alpha_{21}^{(02)}] \equiv R_2^{(1)},
\end{aligned}
$$

or

$$
\alpha_6^{(1)} = \frac{R_1^{(1)} \gamma_2^{(3)} - R_2^{(1)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}, \qquad \alpha_8^{(1)} = \frac{R_2^{(1)} - R_1^{(1)} \gamma_2^{(2)}}{\gamma_2^{(3)} - \gamma_2^{(2)}}.
$$

Since $\alpha_{11}^{(20)}$, $\alpha_{11}^{(02)}$ and $\alpha_{21}^{(20)}$, $\alpha_{21}^{(02)}$ are conjugate pairs and $\alpha_{11}^{(11)}$ and $\alpha_{21}^{(11)}$ are real, $R_1^{(1)}$ and $R_2^{(1)}$ are likewise real. But as $\gamma_2^{(2)}$ and $\gamma_2^{(3)}$ are conjugates, it follows that $\alpha_6^{(1)}$ and $\alpha_8^{(1)}$ are also conjugates.

In order to unify the notation, as at step 1, we put

$$
\begin{aligned}
\alpha_6^{(1)} &= \alpha_{11}^{(10)}, & -\beta_2^{(2)} \alpha_6^{(1)} &= \beta_{11}^{(10)}, & \gamma_2^{(2)} \alpha_6^{(1)} &= \alpha_{21}^{(10)}, & -\delta_2^{(2)} \alpha_6^{(1)} &= \beta_{21}^{(10)},\\
\alpha_8^{(1)} &= \alpha_{11}^{(01)}, & -\beta_2^{(3)} \alpha_8^{(1)} &= \beta_{11}^{(01)}, & \gamma_2^{(3)} \alpha_8^{(1)} &= \alpha_{21}^{(01)}, & -\delta_2^{(2)} \alpha_8^{(1)} &= \beta_{21}^{(01)},
\end{aligned}
$$

and the desired solutions at this step take the form

$$
(26)\qquad
\begin{aligned}
u_{i1} &= \sum_{j,\,k=0}^{2} \alpha_{i1}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t},\\
v_{i1} &= \sum_{j,\,k=0}^{2} \beta_{i1}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j+k = 1 \text{ or } 2, \quad i = 1, 2).
\end{aligned}
$$

If we put

$$\lambda_2 = \mu + \sqrt{-1}\,\nu, \qquad \lambda_3 = \mu - \sqrt{-1}\,\nu,$$

and if we suppose that

$$(27)\qquad \begin{aligned} \alpha_{i1}^{(jk)} &= \frac{1}{2}\,[a_{i1}^{(jk)} + \sqrt{-1}\, b_{i1}^{(jk)}], \quad \alpha_{i1}^{(kj)} = \frac{1}{2}\,[a_{i1}^{(jk)} - \sqrt{-1}\, b_{i1}^{(jk)}],\\ \beta_{i1}^{(jk)} &= \frac{1}{2}\,[c_{i1}^{(jk)} + \sqrt{-1}\, d_{i1}^{(jk)}], \quad \beta_{i1}^{(kj)} = \frac{1}{2}\,[c_{i1}^{(jk)} - \sqrt{-1}\, d_{i1}^{(jk)}] \quad (i = 1, 2), \end{aligned}$$

where $b_{i1}^{(jk)}$ and $d_{i1}^{(jk)}$ are different from zero when $j \neq k$, but equal to zero when $j = k$, then the solutions (26) become

$$u_{i1} = e^{-\mu t}\,[a_{i1}^{(10)} \cos \nu t + b_{i1}^{(10)} \sin \nu t] + e^{-2\mu t}\left[\frac{1}{2}\, a_{i1}^{(11)} + a_{i1}^{(20)} \cos 2\nu\, t + b_{i1}^{(20)} \sin 2\nu\, t\right] \quad (i = 1, 2),$$

$$v_{i1} = e^{-\mu t}\,[c_{i1}^{(10)} \cos \nu\, t + d_{i1}^{(10)} \sin \nu\, t] + e^{-2\mu t}\left[\frac{1}{2}\, c_{i1}^{(11)} + c_{i1}^{(20)} \cos 2\nu\, t + d_{i1}^{(20)} \sin 2\nu\, t\right].$$

The succeeding steps of the integration are similar to the preceding step, and an induction to the general term will now be made to show that the process of integration may be carried on for any desired number of steps.

Let us suppose that u_{il}, v_{il}, $i = 1, 2$, have been computed for $l = 0, \ldots, n-1$, and that

$$(28)\qquad \begin{aligned} u_{il} &= \sum_{j,k=0}^{l+1} \alpha_{il}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t},\\ v_{il} &= \sum_{j,k=0}^{l+1} \beta_{il}^{(jk)}\, e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j + k = 1, 2, \ldots, l+1). \end{aligned}$$

where $\alpha_{il}^{(jk)}$ and $\alpha_{il}^{(kj)}$; $\beta_{il}^{(jk)}$ and $\beta_{il}^{(kj)}$ are conjugate complex if $j \neq k$ but real if $j = k$. In order to make the induction it will be necessary to show from the differential equations in u_{in} and v_{in} obtained from (8′) that the solutions for these variable are the same as (28) if $l = n$.

Step $n+1$: *Coefficients of* ϵ^n *in* (8′). Let us consider the differential equations obtained by equating the coefficients of ϵ^n in (8′) after the various solutions in (28) have been substituted. They are found to have the same form as (22) if the

second subscript on the variables and the right members is changed to n. The right members, however, have the form

$$(29)\quad \begin{aligned} U_{in} &= \sum_{j,k=0}^{n+1} A_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\ V_{in} &= \sum_{j,k=0}^{n+1} B_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \qquad (j + k = 2, 3, \ldots, n + 1; \quad i = 1, 2), \end{aligned}$$

where $A_{in}^{(jk)}$ and $B_{in}^{(jk)}$ are complex quantities similar to those in (28). The complementary functions of these differential equations which have the desired exponentials are the same as (16) if the second subscript on the variables and the superscript on the constants of integration are replaced by n. The particular integrals of these differential equations can be found as at the previous step by the method of the variation of parameters. They are

$$(30)\quad \begin{aligned} u_{1n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{1n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]} e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \alpha_{1n}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\ v_{1n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{2n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]} e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \beta_{1n}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\ u_{2n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{3n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]} e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \alpha_{2n}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\ v_{2n} &= \sum_{j,k=0}^{n+1} \frac{\Delta_{4n}^{(j+k)}}{\Delta[-(j\lambda_2 + k\lambda_3)]} e^{-(j\lambda_2 + k\lambda_3)t} \equiv \sum_{j,k=0}^{n+1} \beta_{2n}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \\ &\qquad (j + k = 2, 3, \ldots, n + 1). \end{aligned}$$

The various Δ's in the preceding equations are defined as follows:

$\Delta[-(j\lambda_2 + k\lambda_3)]$ denotes the determinant Δ in (10) if D is replaced by $-(j\lambda_2 + k\lambda_3)$. Since $j + k$ is not less than 2, these denominators do not become $\Delta(-\lambda_2)$ or $\Delta(-\lambda_3)$ and are therefore different from zero.

$\Delta_{in}^{(j+k)}$, $i = 1, 2, 3, 4$, denote the preceding determinant if the elements of the ith row, reading from top to bottom, are replaced by $A_{1n}^{(jk)}$, $B_{1n}^{(jk)}$, $A_{2n}^{(jk)}$ and $B_{2n}^{(jk)}$, respectively.

On determining the constants of integration $\alpha_6^{(n)}$ and $\alpha_8^{(n)}$ by the initial conditions (18), and then unifying the notation as at step 2, the complete solutions

which have the desired form are found to be the same as (28) if $l = n$. This completes the induction.

The solutions, therefore, at the general step n are

$$
\begin{aligned}
u_{in} &= \sum_{j,\,k=0}^{n+1} \alpha_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}, \\
v_{in} &= \sum_{j,\,k=0}^{n+1} \beta_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t}
\end{aligned} \tag{31}
$$

$$(i = 1, 2; \quad j + k = 1, 2, \ldots, n + 1; \quad n = 0, 1, 2, \ldots).$$

When equations (31) are substituted in (15), the asymptotic solutions of the differential equations (8) are then found to be

$$
\begin{aligned}
\epsilon u_i &= \sum_{n=1}^{\infty} \sum_{j,\,k=0}^{n} \alpha_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \epsilon^n, \\
\epsilon v_i &= \sum_{n=1}^{\infty} \sum_{j,\,k=0}^{n} \beta_{in}^{(jk)} e^{-(j\lambda_2 + k\lambda_3)t} \epsilon^n \qquad (i = 1, 2; \quad j + k = 1, 2, \ldots, n),
\end{aligned} \tag{32}
$$

where the second subscript on the constants $\alpha_{in}^{(jk)}$ and $\beta_{in}^{(jk)}$ has been made to conform with the powers of ϵ.

If these solutions are expressed in trigonometric form by the substitution used in (27) we obtain

$$
\begin{aligned}
\epsilon u_i = f_i\,(+\sqrt{3}, -t) \equiv \sum_{n=1}^{\infty} \sum_{j=1}^{\frac{n}{2},\,\frac{n+1}{2}}{}' \sum_{k=0\text{ or }1}^{j}{}^{*} \Big[\{ a_{in}^{(2j-1,\,2k-1)} &\cos(2k-1)\,\nu t \\
+ b_{in}^{(2j-1,\,2k-1)} \sin(2k-1)\,\nu t \} \, e^{-(2j-1)\mu t} &+ \{ a_{in}^{(2j,\,2k)} \cos 2k\,\nu t \\
&+ b_{in}^{(2j,\,2k)} \sin 2k\,\nu t \} \, e^{-2j\mu t} \Big] \epsilon^n, \\
\epsilon v_i = g_i\,(+\sqrt{3}, -t) \qquad\qquad &(i, = 1, 2),
\end{aligned} \tag{33}
$$

where Σ' denotes that the highest value of $2l - 1$ or $2l$ is n, and Σ^* denotes that the lowest values of $2k - 1$ and $2k$ are 1 and 0, respectively. The function $g_i\,(+\sqrt{3}, -t)$ is the same as $f_i\,(+\sqrt{3}, -t)$ if the constants a and b with their subscripts and superscripts are replaced by c and d, respectively, with the same subscripts and superscripts. When these equations are substituted in (7) we obtain the parametric equations of the orbits which approach the vertices of the equilateral triangles in configuration I, viz.,

$$
(34)\quad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(+\sqrt{3}, -t),\\
y_1 &= \frac{\sqrt{3}}{2}(m_2 + m_3) + g_1(+\sqrt{3}, -t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(+\sqrt{3}, -t),\\
y_2 &= -\frac{\sqrt{3}}{2}m_1 + g_2(+\sqrt{3}, -t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

The orbits which approach the configuration II are obtained by changing the sign of $\sqrt{3}$ in (34). They have the equations

$$
(35)\quad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(-\sqrt{3}, -t),\\
y_1 &= -\frac{\sqrt{3}}{2}(m_2 + m_3) + g_1(-\sqrt{3}, -t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(-\sqrt{3}, -t),\\
y_2 &= \frac{\sqrt{3}}{2}m_1 + g_2(-\sqrt{3}, -t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

Equations (34) and (35) therefore represent the orbits which approach the vertices of the equilateral triangles in configurations I and II, respectively, as the time approaches plus infinity. These orbits deal with the *future* of the system and, at the risk of being censored, we shall cite them as *future* I and *future* II, respectively.

6. **The asymptotic solutions in $e^{\lambda_2 t}$ and $e^{\lambda_3 t}$.** This section deals with the asymptotic orbits which approach the vertices of the equilateral triangles as the time approaches $-\infty$. Such orbits deal with the *past* while the previous orbits forecast the *future*. So then the mathematical astronomer can say with more than *poetic license*

"Backward, turn backward, O time in your flight."

It is obvious that these orbits can be constructed by making the same use of the exponentials $e^{\lambda_3 t}$ and $e^{\lambda_4 t}$ as was made of $e^{-\lambda_3 t}$ and $e^{-\lambda_4 t}$ in constructing the previous orbits. This construction will not be considered in detail as we shall show that the *past* orbits can be obtained from the *future* orbits in a very simple way. This method, it will be observed, is the converse of the adage of "history repeating itself."

Let us consider the differential equations (8) and let their solutions (33) be denoted by $f_i(+\sqrt{3}, -t)$, $g_i(+\sqrt{3}, -t)$, $i = 1, 2$. Next consider the effect of changing the signs of t, $\sqrt{3}$, v_i in (8). The left members of the first and third equations of (8) remain unchanged while those of the second and fourth change signs. Since $u_1^{(k)}$ and $v_1^{(k)}$ are even in $\sqrt{3}$, v_1, and v_2, considered together, the above changes of signs will leave these expressions unaltered. But as the right members of the second and fourth equations in (8) contain the factor $\sqrt{3}$, these changes of signs will produce a change of sign not only in the left members but also in the right members of these two equations and the minus sign can be cancelled off in both equations. Hence the differential equations (8) are unchanged if the signs of t, $\sqrt{3}$, v_1 and v_2 are changed. These changes of signs have no effect upon the initial conditions (18) and consequently if we make the same changes of signs in the solutions $f_i(+\sqrt{3}, -t)$, $g_i(+\sqrt{3}, -t)$, it will still leave them solutions of (8). Thus $f_i(-\sqrt{3}, +t)$ and $-g_i(-\sqrt{3}, +t)$ are solutions of (8) and the corresponding solutions of (1) are

$$
(36)\qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(-\sqrt{3}, +t),\\
y_1 &= \frac{\sqrt{3}}{2}(m_2 + m_3) - g_1(-\sqrt{3}, +t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(-\sqrt{3}, +t),\\
y_2 &= -\frac{\sqrt{3}}{2}m_1 - g_2(-\sqrt{3}, +t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

These are the orbits which approach the vertices of the equilateral triangle in configuration I as the time approaches $-\infty$, and will be cited as *past* I. The corresponding orbits for configuration II, *past* II, are obtained from (36) by changing the sign of $\sqrt{3}$. They are

$$
(37)\qquad
\begin{aligned}
x_1 &= \frac{1}{2}(m_3 - m_2) + f_1(+\sqrt{3}, +t),\\
y_1 &= -\frac{\sqrt{3}}{2}(m_2 + m_3) - g_1(+\sqrt{3}, +t),\\
x_2 &= \frac{1}{2}m_1 + m_3 + f_2(+\sqrt{3}, +t),\\
y_2 &= +\frac{\sqrt{3}}{2}m_1 - g_2(+\sqrt{3}, +t),\\
x_3 &= -\frac{1}{m_3}(m_1 x_1 + m_2 x_2),\\
y_3 &= -\frac{1}{m_3}(m_1 y_1 + m_2 y_2).
\end{aligned}
$$

From the form of the equations of these four orbits it is evident that the *past* II orbits are obtained from the *future* I orbits by changing the sign of t in the latter and reflecting in the x-axis. The same relation exists between *past* I and *future* II orbits.

7. **The convergence of solutions.** Only the formal construction of the solutions (8) has been made, and we shall now consider their convergence.

The convergence of the solutions (32) of the differential equations (8) depends upon the form of the characteristic exponents in terms of which the solutions are expanded. Now it has been shown by Poincaré* that such solutions as (32), (34) and (35) will converge as t approaches $+\infty$ provided that the real parts of the characteristic exponents in terms of which the solutions have been expanded are different from zero and positive, and likewise solutions such as (36) and (37) will converge as t approaches $-\infty$ provided that the corresponding exponents are different from zero and negative. The characteristic exponents of this problem are $0, 0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3$, equations (13), but only λ_2 and λ_3 have their real parts different from zero, and it is in terms of only these exponents that the solutions have been expanded. Hence the solutions (34) and (35) converge as t approaches $+\infty$, and the solutions (36) and (37) converge as t approaches $-\infty$.

8. **Illustrative examples.** We shall conclude this paper with illustrative numerical examples. The values of the masses chosen are not the ratios of the masses of the Sun, Jupiter and the planetoids mentioned in § 1, but more simple values to illustrate the nature of the orbits.

Let $m_1 = 0.2$, $m_2 = 0.3$ and $m_3 = 0.5$. Then M, the sum of the masses, is unity, and n, the mean angular motion, is $+1$ or -1. We have chosen the

* *Loc. cit.*, vol. 1, pp. 338–343.

counter-clockwise direction of rotation and put $n = +1$. The vertices of the equilateral triangle for configuration I are $(0.1,\ 0.4\sqrt{3})$, $(0.6,\ -0.1\sqrt{3})$ and $(-0.4,\ -0.1\sqrt{3})$, and for configuration II $(0.1,\ -0.4\sqrt{3})$, $(0.6,\ 0.1\sqrt{3})$ and $(-0.4,\ 0.1\sqrt{3})$ for m_1, m_2 and m_3, respectively.

The computation has been carried out for only the linear terms in ϵ for the orbits (32). The values of the various constants together with the equations which define them are listed in the following table. The conjugates of the various terms are omitted.

TABLE I

Constant	Equation	Value
Δ	(11)	$D^2(D^6+2D^4+3.0925D^2+2.0925)$
λ_1	(12)	$\sqrt{-1}$
$\lambda_2 = \mu + \sqrt{-1}\,\nu$	(12)	$0.688+0.987\sqrt{-1}$
$\beta_2^{(2)}$	(14) et seq.	$0.152+0.948\sqrt{-1}$
$\gamma_2^{(2)}$	(14) et seq.	$-0.246+0.547\sqrt{-1}$
$\delta_2^{(2)}$	(14) et seq.	$-0.630-0.315\sqrt{-1}$
$\alpha_6^{(0)} = \alpha_{10}^{(10)}$	(16), (19)	$0.500-(0.225+0.914\gamma)\sqrt{-1}$
$-\beta_2^{(2)}\alpha_6^{(0)} = \beta_{10}^{(10)}$	(16), (19)	$-(0.289+0.866\gamma)-(0.440-0.139\gamma)\sqrt{-1}$
$\gamma_2^{(2)}\alpha_6^{(0)} = \alpha_{20}^{(10)}$	(16), (19)	$0.500\gamma+(0.328+0.225\gamma)\sqrt{-1}$
$-\delta_2^{(2)}\alpha_6^{(0)} = \beta_{20}^{(10)}$	(16), (19)	$(0.386+0.288\gamma)+(0.015-0.578\gamma)\sqrt{-1}$

The solutions for ϵu_1, ϵv_1, ϵu_2, and ϵv_2 are

$$
\begin{aligned}
\epsilon u_1 &= e^{-0.688t}\,[\qquad\qquad\qquad\quad \cos 0.987t - (0.450 + 1.828\gamma)\sin 0.987t]\,\epsilon + \cdots,\\
\epsilon v_1 &= e^{-0.688t}\,[(0.578 + 1.732\gamma)\cos 0.987t + (0.880 - 0.278\gamma)\sin 0.987t]\,\epsilon + \cdots,\\
\epsilon u_2 &= e^{-0.688t}\,[\qquad\qquad\qquad\quad \gamma\cos 0.987t + (0.656 + 0.450\gamma)\sin 0.987t]\,\epsilon + \cdots,\\
\epsilon v_2 &= e^{-0.688t}\,[(0.772 + 0.576\gamma)\cos 0.987t + (0.030 - 1.152\gamma)\sin 0.987t]\,\epsilon + \cdots.
\end{aligned}
$$

Example I.

If we put $\epsilon = \gamma = 0.1$ and consider only the linear terms in ϵ, these solutions become

$$
\begin{aligned}
\epsilon u_1 &= \,e^{-0.688t}\,[0.1000\cos 0.987t - 0.0633\sin 0.987t],\\
\epsilon v_1 &= -\,e^{-0.688t}\,[0.0751\cos 0.987t + 0.0852\sin 0.987t],\\
\epsilon u_2 &= \,e^{-0.688t}\,[0.0100\cos 0.987t + 0.0701\sin 0.987t],\\
\epsilon v_2 &= \,e^{-0.688t}\,[0.0830\cos 0.987t - 0.0085\sin 0.987t].
\end{aligned}
$$

The values of the above quantities for various values of t are listed in Table II. The corresponding values for ϵu_3 and ϵv_3 have been computed from the center

of gravity equations. The coördinates ($\epsilon\, u_1$, $\epsilon\, v_1$), ($\epsilon\, u_2$, $\epsilon\, v_2$) and ($\epsilon\, u_3$, $\epsilon\, v_3$) denote the x- and y-displacements of m_1, m_2 and m_3, respectively, from the vertices of the equilateral triangle in configuration I.

TABLE II

$\epsilon = 0.1 \quad \gamma = 0.1$

t	ϵu_1	ϵv_1	ϵu_2	ϵv_2	ϵu_3	ϵv_3
0	+0.100	−0.075	+0.010	+0.083	−0.046	−0.020
0.1	+0.087	−0.078	+0.016	+0.076	−0.044	−0.015
0.2	+0.075	−0.079	+0.021	+0.070	−0.042	−0.010
0.3	+0.063	−0.079	+0.024	+0.063	−0.040	−0.006
0.4	+0.052	−0.078	+0.028	+0.056	−0.037	−0.002
0.5	+0.041	−0.076	+0.030	+0.049	−0.034	+0.001
0.6	+0.032	−0.073	+0.031	+0.043	−0.032	+0.004
0.7	+0.023	−0.069	+0.032	+0.036	−0.029	+0.006
0.8	+0.015	−0.065	+0.033	+0.030	−0.026	+0.008
0.9	+0.008	−0.061	+0.033	+0.025	−0.023	+0.010
1	+0.001	−0.057	+0.032	+0.019	−0.020	+0.011
1.2	−0.009	−0.047	+0.030	+0.010	−0.014	+0.013
1.4	−0.017	−0.037	+0.027	+0.003	−0.010	+0.013
1.6	−0.021	−0.028	+0.023	−0.003	−0.005	+0.013
1.8	−0.024	−0.020	+0.019	−0.007	−0.002	+0.012
2	−0.025	−0.012	+0.015	−0.010	+0.001	+0.011
2.4	−0.022	−0.001	+0.008	−0.013	+0.004	+0.008
2.8	−0.017	+0.006	+0.002	−0.012	+0.005	+0.005
3.2	−0.011	+0.008	−0.001	−0.009	+0.005	+0.002
3.6	−0.006	+0.009	−0.003	−0.006	+0.004	+0.0002
4	−0.0015	+0.0072	−0.0037	−0.0033	+0.0028	−0.0009
4.5	+0.0015	+0.0046	−0.0032	−0.0006	+0.0011	−0.0015
5	+0.0027	+0.0021	−0.0021	+0.0008	+0.0002	−0.0014
5.5	+0.0026	+0.0003	−0.0011	+0.0014	−0.0004	−0.00097
6	+0.0019	+0.0006	−0.0003	+0.0013	−0.0006	−0.00052
7	+0.0004	+0.0009	+0.0004	+0.0005	−0.0004	+0.00005

The diagram of the orbit of m_1 for Example I is shown in Figure 3. This orbit is, of course, with respect to the rotating axes. The arrow indicates the directions of motion.

Example II.

We have also computed the orbits when $\epsilon = 0.1$ and $\gamma = 1$. The equations determining the displacements, in so far as the linear terms in ϵ are concerned, are

$$\begin{aligned}
\epsilon u_1 &= e^{-0.688t}\,(0.100 \cos 0.987t - 0.228 \sin 0.987t),\\
\epsilon v_1 &= -e^{-0.688t}\,(0.231 \cos 0.987t + 0.060 \sin 0.987t),\\
\epsilon u_2 &= e^{-0.688t}\,(0.100 \cos 0.987t + 0.111 \sin 0.987t),\\
\epsilon v_2 &= e^{-0.688t}\,(0.135 \cos 0.987t - 0.112 \sin 0.987t).
\end{aligned}$$

On substituting in these equations the various values of t as in Table II and making use of the center of gravity equations to determine $\epsilon\, u_3$ and $\epsilon\, v_3$, the following displacements are found:

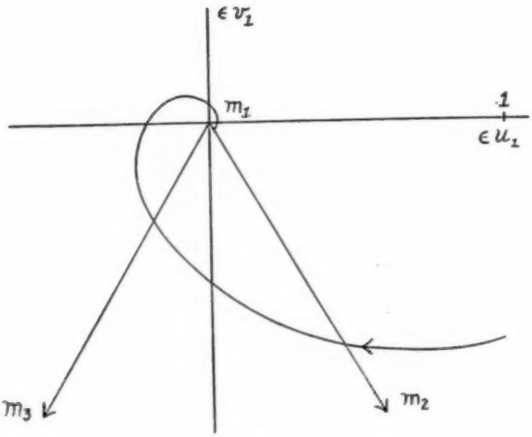

FIG. 3.

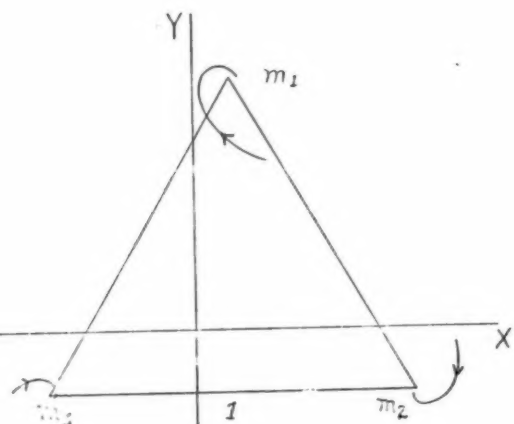

FIG. 4.

TABLE III

$\epsilon = 0.1 \quad \nu = 1$

t	ϵu_1	ϵv_1	ϵu_2	ϵv_2	ϵu_3	ϵv_3
0	+0.100	−0.231	+0.100	+0.135	−0.100	+0.011
0.1	+0.072	−0.221	+0.103	+0.115	−0.091	+0.019
0.2	+0.046	−0.208	+0.105	+0.096	−0.081	+0.026
0.3	+0.024	−0.195	+0.104	+0.079	−0.072	+0.031
0.4	+0.004	−0.179	+0.103	+0.062	−0.063	+0.034
0.5	−0.014	−0.164	+0.100	+0.047	−0.054	+0.038
0.6	−0.029	−0.149	+0.096	+0.033	−0.046	+0.040
0.7	−0.042	−0.133	+0.092	+0.020	−0.038	+0.041
0.8	−0.053	−0.119	+0.086	+0.008	−0.031	+0.042
0.9	−0.061	−0.104	+0.080	−0.001	−0.024	+0.042
1.0	−0.068	−0.090	+0.074	−0.010	−0.018	+0.041
1.2	−0.076	−0.063	+0.062	−0.023	−0.007	+0.039
1.4	−0.078	−0.039	+0.049	−0.032	+0.002	+0.035
1.6	−0.076	−0.019	+0.037	−0.038	+0.009	+0.030
1.8	−0.071	−0.003	+0.026	−0.040	+0.013	+0.025
2	−0.063	+0.009	+0.016	−0.040	+0.016	+0.020
2.4	−0.044	+0.024	+0.001	−0.034	+0.017	+0.011
2.8	−0.026	+0.028	−0.008	−0.024	+0.015	+0.003
3.2	−0.011	+0.026	−0.011	−0.015	+0.011	−0.001
3.6	−0.0001	+0.0198	−0.0114	−0.0066	+0.0069	−0.0039
4	+0.0061	+0.0130	−0.0095	−0.0008	+0.0033	−0.0047
4.5	+0.0087	+0.0054	−0.0061	+0.0032	+0.0001	−0.0041
5	+0.0078	+0.0003	−0.0028	+0.0045	−0.0015	−0.0028
5.5	+0.0054	−0.0024	−0.0004	+0.0039	−0.0019	−0.0014
6	+0.0028	−0.0031	+0.0009	+0.0027	−0.0017	−0.0004
7	−0.0004	−0.0018	+0.0012	+0.0004	−0.0005	−0.0005

The diagram of the orbits of the three bodies in this example is found in Figure 4. The length of the side of the equilateral triangle is taken as the unit length.

In conclusion, the author wishes to express his thanks to his former colleague, F. M. Wood, M.A., B.Sc., for verifying the algebraic and numerical computations in this paper.

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ERRATA, VOLUME 23

Page 51. J. F. RITT, *Prime and composite polynominals.*

Page 63. Line 7, for $\psi^{-1}(v)$ read $\psi^{-1}(w)$;

line 25, for $\varphi^{-1}(v)$ read $\psi^{-1}(w)$.

TRANSACTIONS

OF THE

American Mathematical Society

EDITED BY

GEORGE DAVID BIRKHOFF

ARTHUR BYRON COBLE

LUTHER PFAHLER EISENHART

Volume 23. Number 4
June 1922

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1923

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Vol. 23. No. 4 **June 1922** **Whole No. 92**

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Published semi-quarterly by the American Mathematical Society, Lancaster, Pa., and 501 W. 116th St., New York City.

Entered as second-class matter at the Post-Office at Lancaster, Pa., under the Act of October 3, 1917.